On the complexity of finding chordless paths in bipartite graphs and some interval operators in graphs and hypergraphs

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December 2009

Abstract

In this paper we show that the problem of finding a chordless path between a vertex s and a vertex t containing a vertex v remains NP-complete in bipartite graphs, thereby strengthening previous results on the same problem. We show a relation between this problem and two interval operators: the simple path interval operator in hypergraphs and the even-chorded path interval operator in graphs. We show that the problem of computing the two mentioned intervals is NP-complete.

Key words: Chordless paths; Bipartite graphs; Interval operators; Graphs convexity; Simple paths; Even-chorded paths

1. Introduction

Given a graph G, a path P of *length* k is a sequence $(v_0, v_1, \ldots, v_k)$, $k \geq 0$ of distinct vertices such that $v_i v_{i+1}$, $1 \leq i < k$ is an edge of G. A *chord* of a path is an edge joining two non consecutive vertices. A path is *chordless* (or *induced*) if it contains no chord. Here we discuss the problem of finding a chordless path between two given vertices s and t containing a specified vertex v in bipartite graphs. This problem, called Cp3v, has been shown to be NP-complete for general graphs [2, 14]. The Cp3v problem arises in the context of service deployment in communication networks [14] and is related to the study of perfect graphs [2, 4, 6, 7, 5].

Let V be the set of vertices of a connected graph or hypergraph. An *interval operator* (also called transit function in [3]) is a function $I : V \times V \to 2^V$ with the property that $\{u, v\} \subseteq I(u, v)$ and $I(u, v) = I(v, u)$ [19]. Usually the interval operator is defined in terms of a family of paths in a graph (or in a hypergraph). Prime example of interval operator is the *geodesic interval* in a graph. It contains every vertex on every shortest path between u and v. Other examples of interval operators are the *monophonic* (or *induced-path*) interval, which contains every vertex on every chordless (induced) path between u and v, and the *all-paths* interval, which is the set of all vertices lying on paths between u and v.

Given an interval operator I, a subset A of V is said *I-convex* if $I(u, v) \subseteq A$ for all u, v in A. The *I-convex hull* of a subset A of V is the smallest I-convex set containing A. Each interval operator defines an *alignment* on V. An alignment is a set $\mathcal{L}$ of subsets of V, that satisfies the following properties :

1. $\emptyset, V \in \mathcal{L}$
2. $X \cap Y \in \mathcal{L}$ for any two elements X and Y of $\mathcal{L}$

the couple $(V, \mathcal{L})$ is called a *convexity space* [19, 16]. The elements of $\mathcal{L}$ are exactly the convex sets. An element p of an I-convex set A is an *extreme point* of A if $A - \{p\}$ is I-convex. If the convexity space satisfies the following property (Minkowski-Krein-Milman property):

Every convex set is the hull of its extreme points

then it is called a *geometric convexity*.

Some works [2, 3, 10, 14, 16] have investigated the computational complexity of interval operators and the complexity of computing the convex hulls for different type of convexities. In [3] a number of specific interval operators (referred to as transit functions) and a list of some basic facts about them are given, while in [2, 10, 14] it is shown that the monophonic interval is computationally hard. An interesting fact is that for some families of paths, computing the interval I is NP-hard, while computing the I-convex hull requires polynomial time. An example of this is the monophonic convexity, whose interval, as we already mentioned, is NP-hard, while computing the monophonic convex hull can be done in polynomial time [10, 16].

Other two instances of interval operators are the *simple path* interval in hypergraphs and the *even-chorded path* interval in graphs. The first one defines the *simple path convexity* (s.p. convexity) in hypergraphs, while the second one defines the *strongly chordal convexity* (s-convexity) in graphs [13]. In [13], both the s. p. convexity and the s-convexity are studied for hypergraphs and graphs (see Section 3 for all the definitions on hypergraphs, simple paths, the simple path and the even-chorded path intervals and their related convexities) and a characterization is given of the hypergraphs and graphs for which the s.p. convexity space and the s-convexity space, respectively, are geometric.

In this paper we show that Cp3v problem in bipartite graphs is related to the computation of the simple path and even-chorded path interval operators.

We discuss the complexity of computing the simple path and the even-chorded path intervals in hypergraphs and graphs, respectively, by showing that the problem of deciding if a vertex v belong to a simple path between two given vertices s and t, hereafter referred as the Spi problem and the problem of deciding if a vertex v belong to an even-chorded path between two given vertices s and t, hereafter referred as the Ecpi problem, are NP-complete. We do this by first demonstrating that the Cp3v problem remains NP-complete in bipartite graphs.

The hypergraphs and graphs for which the s.p. convexity space and the s-convexity space are convex geometries turn out to be correlated to each other by the concept of *totally balanced hypergraph* [12] (see also Section 3). The study of totally balanced hypergraphs and the study of algorithmic aspects related to the two above mentioned convexities, has application, for example, in database theory [1, 8, 11, 15, 16, 17, 18].

The work is organized as follows. In Section 2 we give the proof that the Cp3v problem is NP-complete in bipartite graphs. In Section 3, using result of the Section 2 we give the proof that the Spi and the Ecpi problems are NP-complete. In Section 4 we give some concluding remarks.

2. The reduction

We will use more or less standard graph theory definitions. Let G be an undirected loopless simple graph. The *vertex set* and the *edge set* of G are denoted by $V(G)$ and $E(G)$, respectively; furthermore $n = |V(G)|$. We say that a path $(v_0, \ldots, v_k)$, $k > 0$, in a graph G contains an edge e of G if $e = v_i v_{i+1}$ for some $0 \leq i < k$. Given two vertices v_i and v_j, $i < j$ of a path P then the *subpath* of P between v_i and v_j is the subsequence $(v_i, v_{i+1}, \ldots, v_{j-1}, v_j)$ of consecutive vertices of P.

Two vertices of G are *adjacent* if they are the end points of an edge. We denote by $N_G(v)$ the set of vertices adjacent to a vertex v. The *closed neighborhood* of a vertex is $N_G[v] = N_G(v) \cup \{v\}$. A set of pairwise non adjacent vertices is an *independent* set. A graph is bipartite if there exists a partition (V_1, V_2) of $V(G)$ such that both V_1 and V_2 are independent sets.
We show that CP3V is NP-complete in bipartite graphs using a reduction from the *independent set* problem. The reduction we present here is somewhat similar to the one given in [14].

The independent set problem asks, given a graph G and an integer k, if there exists an independent set of size at least k of vertices of G.

We shall construct a bipartite graph G' such that there exists a chordless path between two vertices s and t containing a vertex v if and only if there exists in G an independent set of vertices of size at least k.

The graph G' is built with two basic structures (partly similar to the structures called in [14] vertex choice diamonds) as shown if Fig. 1.

The structure of type (a) has two vertices s^h and t^h and vertices $v_1^h, v_2^h, \ldots, v_n^h$. Each vertex v_i^h is connected to the vertices s^h and t^h. We make a copy of each structure of type (a) and in this copy we denote the vertices s^h, t^h and v_i^h as p^h, q^h and w_i^h respectively, $i = 1, \ldots, n$ and $h = 1, \ldots, k$.

We connect together all these structures by identifying t^h with s^{h+1} and by identifying p^h with q^{h+1} for $1 \leq h < k$. We also add the edge $t^k p^1$.

The structure of type (b) has two vertices σ^h and τ^h and vertices $v_1^{h,1}, v_1^{h,2}, v_1^{h,3}, v_2^{h,1}, v_2^{h,2}, v_2^{h,3}, \ldots, v_n^{h,1}, v_n^{h,2}, v_n^{h,3}$. Each vertex $v_i^{h,1}$ is connected to the vertex $v_i^{h,2}$ and each vertex $v_i^{h,2}$ is connected to the vertex $v_i^{h,3}$, $i = 1, \ldots, n$. All the vertices $v_i^{h,1}$ are connected to σ^h and all the vertices $v_i^{h,3}$ are connected to τ^h, $i = 1, \ldots, n$. We make a copy of each structure of type (b) and in this copy we denote the vertices σ^h, τ^h and $v_i^{h,j}$ as π^h, θ^h and $\omega_i^{h,j}$ respectively, $j = 1, 2, 3$, $i = 1, \ldots, n$ and $h = 1, \ldots, k$.

We connect together all these structures by identifying τ^{h-1} with σ^h and by identifying θ^{h-1} with π^h for $1 < h < k$. We also add the edge $\tau^k \pi^1$.

Then we add the edge $\theta^k q^k$.

For all $h = 1, \ldots, k$ and for all $i \neq j$ we add the following edges

1. $v_i^h v_j^{h,1}$
2. $v_i^h v_j^{h,3}$
3. $v_i^h \omega_j^{h,2}$
4. $v_i^h w_j^h$
5. $w_i^h \omega_j^{h,3}$
6. $w_i^h \omega_j^{h,1}$
7. $w_i^h v_j^{h,2}$

8. $\omega_i^{h,2}\nu_j^{h,2}$

and we call them *consistency edges of type* (t), $t = 1, \dots, 8$. For short, we refer to them as $C(t)$ edges, $t = 1, \dots, 8$.

We add to G' the edge $v_i^h w_i^r$ and the edge $\nu_i^{h,2} w_i^r$ for all $1 < h \leq k$ and for all $1 \leq r < h$, $i = 1, \dots, n$. These edges are called *set edges*.

For each edge $v_i v_j$ of G we add to G' the edge $v_i^h w_j^r$ and the edge $\nu_i^{h,2} w_j^r$ for all $1 < h \leq k$ and for all $1 \leq r < h$, $i = 1, \dots, n$. These edges are called *independent edges*.

Example 1. *In Fig. 2 is shown an example of a graph G and the corresponding graph G'. In Fig. 3 only consistency edges from type 1 to type 4 are drawn and in Fig. 4 only consistency edges from type 5 to type 8 are drawn. In Fig. 5 only set and independent edges are drawn.*

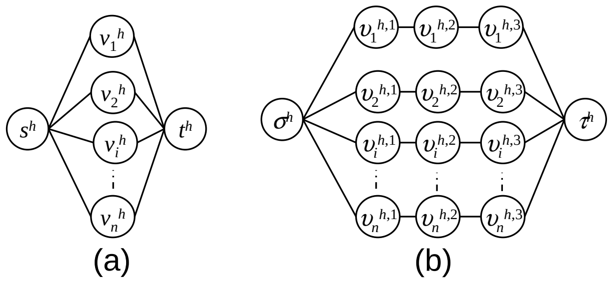

Figure 1: The basic structures

Remark 1. *The graph G' is bipartite.*

Observe that given a graph G and an integer k the graph G' has $8nk + 4k + 4$ vertices. Therefore

Remark 2. *Given a graph G and an integer k the graph G' can be constructed in polynomial time.*

First of all we prove the following

Lemma 1. *Let P be a chordless path between s^1 and σ^1 containing q^k. Then P contains t^k and τ^k. Furthermore P includes the subpaths $(s^1, v_{i_1}^1, t^1, \dots, s^k, v_{i_k}^k, t^k)$ and $(\tau^k, \nu_{i_k}^{k,3}, \nu_{i_k}^{k,2}, \nu_{i_k}^{k,1}, \sigma^k, \dots, \tau^1, \nu_{i_1}^{1,3}, \nu_{i_1}^{1,2}, \nu_{i_1}^{1,1}, \sigma^1)$ which contain no consistency no set nor independent edges.*

Proof: We prove the Lemma by induction on $m = 1, \dots, k$.
Basis $m = 1$. Let P be a chordless path from s^1 to σ^1 containing q^k. Let v_i^1 be the first vertex of P after s^1. Suppose that a $C(1)$ edge $v_i^1 \nu_j^{1,1}$ is in P. Then $\nu_j^{1,1}\sigma^1$ would be a chord of P. Therefore no $C(1)$ edge of the form $v_i^1 \nu_j^{1,1}$ is in P and P must contain $\nu_i^{1,1}$. Also note that P cannot contain any $C(1)$ edge of the form $\nu_i^{1,1} v_j^1$ for, otherwise, $s^1 v_j^1$ would be a chord of P. It follows that $\nu_i^{1,2}$ must also be in P.

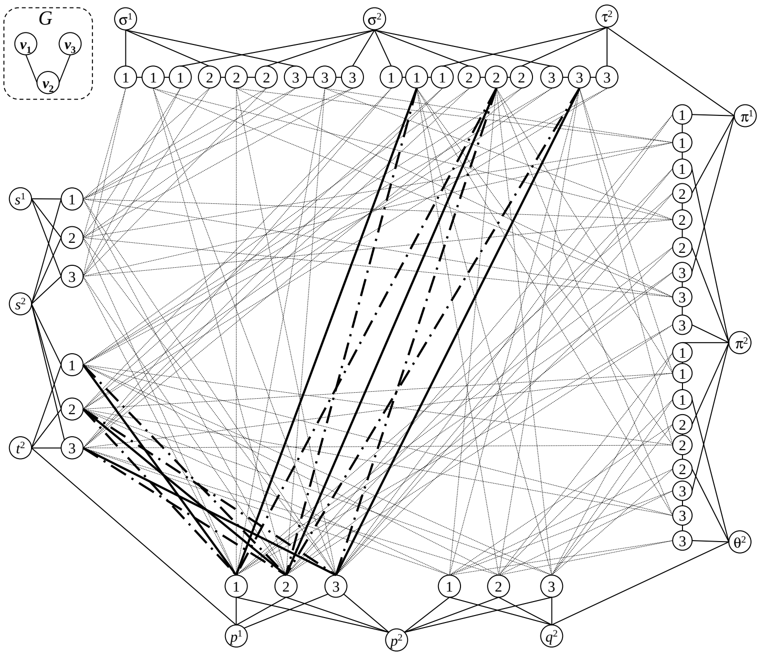

Figure 2: An example of transformation with $k = 2$. In the inset above on the left the graph G. The consistency edges are shown in dashed lines. The set edges are shown in solid bold lines. The independent edges are shown in a mixed dotted an dashed lines.

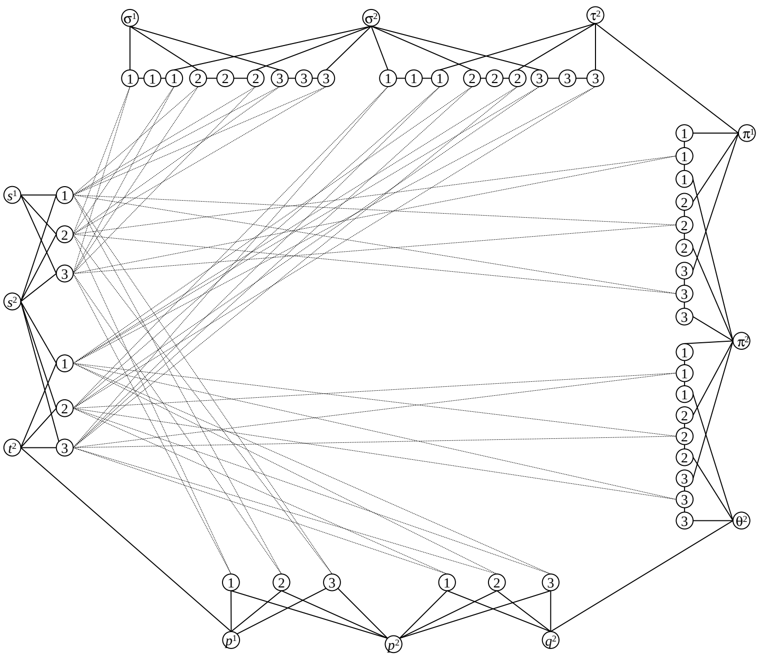

Figure 3: The consistency edges from type 1 to 4 of Example 1.

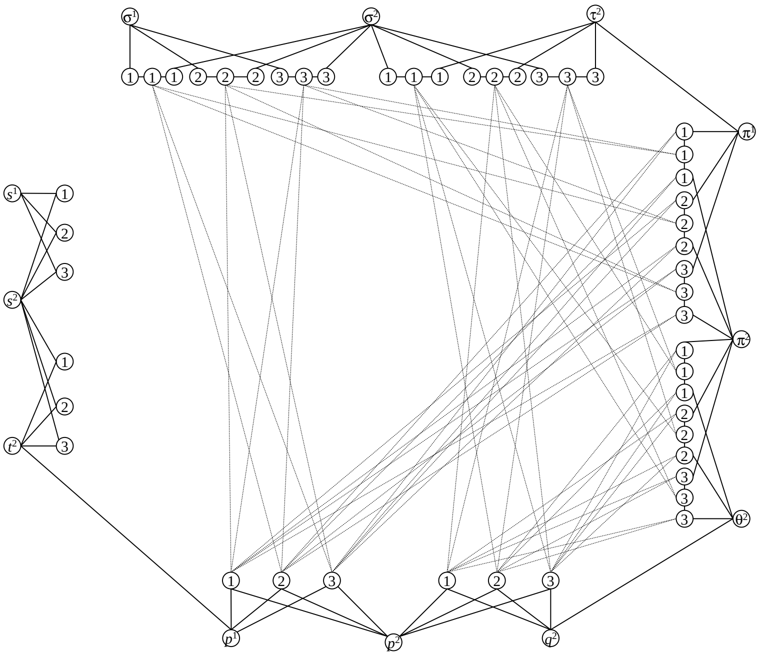

Figure 4: The consistency edges from type 5 to 8 of Example 1.

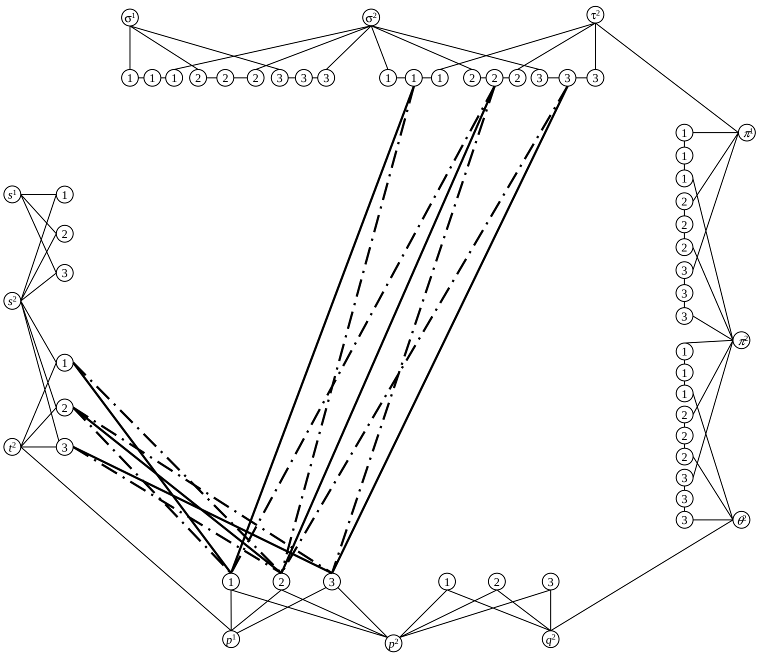

Figure 5: The set and independent edges of Example 1.

If a $C(7)$ edge $v_i^{1,2}w_j^1$ or a $C(8)$ edge $v_i^{1,2}\omega_j^{1,2}$ is in P then the $C(4)$ edge $w_j^1v_i^1$ or the $C(3)$ edge $\omega_j^{1,2}v_i^1$ would be, respectively, a chord of P. From this follows that the edge $v_i^{1,2}v_i^{1,3}$ must be in P. At this point we note that no $C(2)$ edge of the form $v_i^{1,3}v_j^1$ is in P for otherwise $s^1v_j^1$ would be a chord of P, and the only possibility is that $v_i^{1,3}\tau^1$ is in P.

Suppose now that a $C(2)$ edge $v_i^1v_j^{1,3}$ is in P. Since τ^1 is in P then $v_j^{1,3}\tau^1$ would be a chord of P (contradiction).

Suppose then that a $C(3)$ edge $v_i^1\omega_j^{1,2}$ or a $C(4)$ edge $v_i^1w_j^1$ is in P. Since P contains $v_i^{1,2}$, then the $C(8)$ edge $\omega_j^{1,2}v_i^{1,2}$ or the $C(7)$ edge $w_j^1v_i^{1,2}$ would be, respectively, a chord of P (contradiction).

Since no set or independent edges are incident to v_i^1 the next vertex of P must be t^1 and the subpaths of P between τ^1 and σ^1 and between s^1 and t^1 contain no consistency no set nor independent edges. This concludes the basis step.

Induction step. Let $1 < m \leq k$. By the induction hypothesis the path P contains $s^m = t^{m-1}$ and $\sigma^m = \tau^{m-1}$ and the subpaths between s^1 and t^{m-1} and between τ^{m-1} and σ^1 contain no consistency no set nor independent edges. Let v_i^m be the vertex of P immediately subsequent of s^m.

Suppose that a $C(1)$ edge $v_i^mv_j^{m,1}$ is in P. Then $v_j^{m,1}\sigma^m$ would be a chord of P. Therefore no $C(1)$ edge of the form $v_i^mv_j^{m,1}$ is in P and P must contain $v_i^{m,1}$. Also note that P cannot contain any $C(1)$ edge of the form $v_i^{m,1}v_j^m$ for otherwise $s^mv_j^m$ would be a chord of P. It follows that $v_i^{m,2}$ must also be in P.

Now let $X = N_{G'}(v_i^{m,2}) - \{v_i^{m,1}, v_i^{m,3}\}$. A vertex x is in X due to a $C(7)$ edge $v_i^{m,2}w_j^m$ or a $C(8)$ edge $v_i^{m,2}\omega_j^{m,2}$ or a set edge $v_i^{m,2}w_i^r$, $1 \leq r < m$ or an independent edge $v_i^{m,2}w_j^r$, $1 \leq r < m$. Each vertex in X is also adjacent to v_i^m due to the presence in G' of a $C(4)$ edge $v_i^mw_j^m$ or a $C(3)$ edge $v_i^m\omega_j^{m,2}$ or a set edge $v_i^mw_i^r$ or an independent edge $v_i^mw_j^r$. Therefore no vertex of X could be in P. From this follows that neither $v_i^{m,2}x$ nor v_i^mx could be in P. As a consequence the edge $v_i^{1,2}v_i^{1,3}$ must be in P (see fig. 6).

At this point we note that no $C(2)$ edge of the form $v_i^{m,3}v_j^m$ is in P for otherwise $s^mv_j^m$ would be a chord of P, and the only possibility is that $v_i^{m,3}\tau^1$ is in P .

Suppose now that a $C(2)$ edge $v_i^mv_j^{m,3}$ is in P. Since τ^m is in P then $v_j^{m,3}\tau^m$ would be a chord of P (contradiction).

Therefore the next vertex of v_i^m must be t^m. By what said above we have that the subpaths of P between τ^m and σ^m and between s^m and t^m contains no consistency no set nor independent edges. This concludes the induction step. $\square$

By Lemma 1, any chordless path between s^1 and σ^1 containing q^k must contain t^k and τ^k. Hence it must contain also p^1 and π^1. Furthermore we have the following

Lemma 2. *Let P be a chordless path between s^1 and σ^1 containing q^k. Then P contains no $C(3)$ and no $C(4)$ edges and for any given $h \in \{1, \ldots, k\}$, P contains at most one vertex among*

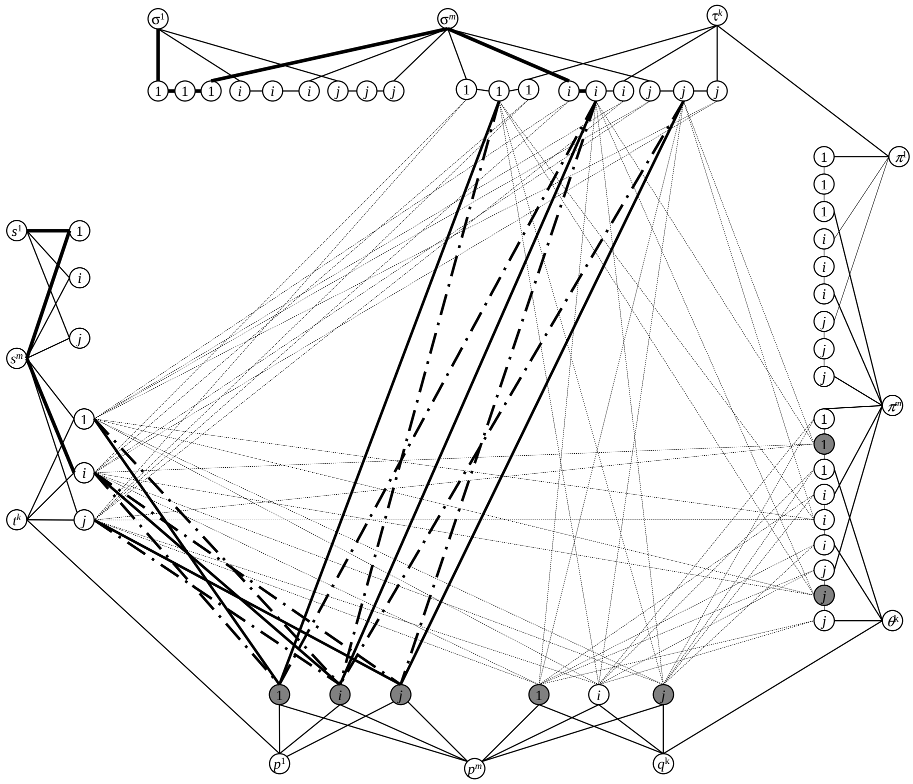

Figure 6: The gray vertices are all adjacent to v_i^m and $v_i^{m,2}$. Therefore no edge of the form $v_i^{m,2}x$ or $v_i^m x$ for all $x \in N_{G'}(v_i^{m,2}) - \{v_i^{m,1}, v_i^{m,3}\}$ could be in P.

$\{w_1^h, \ldots, w_n^h\}$.

Proof: By Lemma 1, P contains the subpath $(s^1, v_{i_1}^1, \ldots, v_{i_k}^k, t^k)$ and neither a $C(3)$ nor a $C(4)$ edge incident to $v_{i_h}^h$, $h = 1, \ldots, k$ is in P. If w_j^h, $j \neq i_h$, is in P then the $C(4)$ edge $v_{i_h}^h w_j^h$ would be a chord of P. Therefore at most one $w_{i_h}^h$ among $\{w_1^h, \ldots, w_n^h\}$ can be in P. Now if a $C(4)$ edge $w_{i_h}^h v_j^h$ is in P then $v_j^h s^h$ would be a chord of P. Analogously we have that if a $C(3)$ edge $\omega_i^{h,2} v_j^h$ is in P then $v_j^h s^h$ would be a chord of P (contradiction). □

Lemma 3. *Let P be a chordless path between s^1 and σ^1 containing q^k. Then P contains θ^k. Furthermore P includes the subpaths $(p^1, w_{i_1}^1, \ldots, w_{i_k}^k, q^k)$ and $(\theta^k, \omega_{i_k}^{k,3}, \omega_{i_k}^{k,2}, \omega_{i_k}^{k,1}, \ldots, \omega_{i_1}^{1,3}, \omega_{i_1}^{1,2}, \omega_{i_1}^{1,1}, \pi^1)$ which contain no consistency no set nor independent edges.*

Proof: We prove the lemma by induction on $m = k, \ldots, 1$ that the subpath between p^m and q^k and the subpath between θ^k and π^m satisfy the conditions of Lemma. Recall that by Lemma 2, no $C(3)$ and $C(4)$ edges are in P.

Basis $m = k$. By hypothesis P is a chordless path and by Lemma 1, there exists a vertex v_i^k in P. By Lemma 2, only w_i^k is in P. Furthermore since $q^k\theta^k$ is in G' then θ^k is in P.

If a $C(5)$ edge $w_i^k\omega_j^{k,3}$ is in P then $\omega_j^{k,3}\theta^k$ would be a chord of P. Therefore no $C(5)$ edge of the form $w_i^k\omega_j^{k,3}$ is in P and we have that $\omega_i^{k,3}$ must be in P. Also note that P cannot contain any $C(5)$ edge of the form $\omega_i^{k,3}w_j^k$ for otherwise $w_j^kq^k$ would be a chord of P. It follows that $\omega_i^{k,2}$ must also be in P.

Note that no $C(8)$ edge $\omega_i^{k,2}v_h^{k,2}$ could be in P for otherwise the $C(7)$ edge $v_h^{k,2}w_i^k$ would be a chord of P. From this follows that the edge $\omega_i^{k,2}\omega_i^{k,1}$ must be in P. At this point we note that no $C(6)$ edge $\omega_i^{k,1}w_j^k$ is in P for otherwise $w_j^kq^k$ would be a chord of P and the only possibility is that $\omega_i^{k,1}\pi^k$ is in P.

Suppose that a $C(6)$ edge $w_i^k\omega_j^{k,1}$ is in P. Since π^k is in P we have that $\pi^k\omega_j^{k,1}$ would be a chord of P (contradiction).

Suppose now that a $C(7)$ edge $w_i^kv_j^{k,2}$ is in P. Since $\omega_i^{k,2}$ is in P then the $C(8)$ edge $v_j^{k,2}\omega_i^{k,2}$ would be a chord of P (contradiction).

Since no set or independent edges are incident to w_i^k it follow that p^k is in P. This proves that the subpaths of P between p^k and q^k and between θ^k and π^k contain no consistency no set nor independent edges.

Induction step. Let $k > m \geq 1$. By the induction hypothesis q^m and θ^m are in P and the subpaths between q^m and q^k and between θ^k and θ^m contain no consistency no set nor independent edges.

Let v_i^m be a vertex of P (which must exist by Lemma 1). By Lemma 2, we have that w_i^m is in P. Note that no set and no independent edge of the form $w_i^mv_j^r$, $1 < m < r \leq k$ could be in P for otherwise, if v_j^r is in P, then this contradict Lemma 1, and if v_j^r is not in P, then $v_j^rs^r$ would be a chord of P.

With an argument similar to the one used in the basis step, we can show that no $C(5)$ edge incident to w_i^m is in P. Furthermore the vertices $\omega_i^{m,3}$, $\omega_i^{m,2}$, $\omega_i^{m,1}$ and π^m must also be in P. The same argument can also be used to show that neither $C(6)$ nor $C(7)$ edge incident to w_i^m is in P.

Suppose that a set edge $w_i^mv_i^{r,2}$ is in P, $m < r \leq k$. Then consider the vertex v_h^r of P (which must exist by Lemma 1). If $h = i$ then the set edge $w_i^mv_h^r$ would be, by Lemma 1, a chord of P. Therefore $h \neq i$. By Lemma 1, and by the induction hypothesis, P must contain w_h^r. But then the $C(7)$ edge $v_i^{r,2}w_h^r$ would be a chord of P. Analogously we can prove, using a similar argument, that no independent edge of the form $w_i^mv_j^{r,2}$ is in P, $m < r \leq k$.

It follow that the vertex preceding w_i^m is p^m. Hence we have that the subpaths of P between p^m and q^k and between θ^k and π^m contain no consistency no set nor independent edges. This concludes the induction step. $\square$

Lemma 4. *Let P be a chordless path between s^1 and σ^1 containing q^k. If P contains $v^j_{i_j}$ then it contains $w^j_{i_j}$ for all $j = 1, \dots, k$.*

Proof: By Lemma 1 and Lemma 3, any chordless path between s^1 and σ^1 containing q^k does not use any consistency, set or independent edges and therefore it contains $s^1, v^1_{i_1}, \dots, v^k_{i_k}, t^k$ and $p^1, w^1_{j_1}, \dots, w^k_{j_k}, q^k$. If $i_h \neq j_h$ for some $1 \leq h \leq k$ then the $C(4)$ edge $v^h_{i_h} w^h_{j_h}$ would be a chord of P (contradiction). □

Theorem 1. Cp3v *is NP-complete in bipartite graphs.*

Proof: It is easy to see that Cp3v is in NP. We shall show that given a graph G and an integer k, there exist in G' a chordless path between s^1 to σ^1 containing q^k if and only if there exists in G an independent set of size at least k. Suppose that there exists a chordless P path between s^1 and σ^1 containing q^k. Let $I = \{v_{i_j} \in G : v^j_{i_j} \in P\}$. First of all no two vertices v^h_i and v^ℓ_i are in P for $1 \leq h < \ell \leq k$. In fact if v^h_i and v^ℓ_i are in P then, by Lemma 4, w^h_i is also in P. Then the set edge $v^\ell_i w^h_i$ would be a chord of P. Therefore the set I has cardinality k. We now show that I is independent in G. Suppose not and let v_{i_j} and v_{i_ℓ} be two vertices of I adjacent in G and let $v^j_{i_j}$ and $v^\ell_{i_\ell}$ be the corresponding vertices of P with $1 \leq j < \ell \leq k$. By Lemma 4, the vertex $w^j_{i_j}$ is contained in P. But then the independent edge $v^\ell_{i_\ell} w^j_{i_j}$ would be a chord of P, a contradiction. Finally it is easy to see that given an independent set of size k there exists in G' a chordless path between s^1 and σ^1 containing q^k. By Remark 1 and by Remark 2, G' is bipartite and can be obtained from G and k in polynomial time. This completes the proof. □

3. Simple path and even-chorded path interval operators

A hypergraph is a family $\mathcal{H}$ of non empty sets whose union, denoted by $V(\mathcal{H})$, is called the *vertex set* of $\mathcal{H}$. Each element of $\mathcal{H}$ is an *(hyper)edge*. A path of length $k \geq 0$, in $\mathcal{H}$ is a sequence $(x_0, e_1, x_1, \dots, e_k, x_k)$ of pairwise distinct vertices x_i and pairwise distinct edges e_i of $\mathcal{H}$ such that $\{x_{i-1}, x_i\} \subseteq e_i$ for $1 \leq i \leq k$. A path in $\mathcal{H}$ is *simple* if $e_i \cap \{x_0, \dots, x_k\} = \{x_{i-1}, x_i\}$ for $1 \leq i \leq k$ [13]. A *simple cycle* in a hypergraph is defined in the same way as a simple path with the exception that the first and the last vertex do coincide and $k \geq 2$. The *length* of a cycle is the number of its distinct vertices. A hypergraph is *totally balanced* if it contains no simple cycle of length greater than two [11, 12]. The *neighborhood hypergraph* of a graph G, denoted by $\mathcal{N}(G)$, is given by $\mathcal{N}(G) = \{N_G[v] : v \in V(G)\}$. The *two-section* of a hypergraph $\mathcal{H}$ is a graph $G_{\mathcal{H}}$ whose vertex set is $V(G_{\mathcal{H}}) = V(\mathcal{H})$ and whose edge set $E(G_{\mathcal{H}})$ contains an edge uv if and only if $\{u, v\} \subseteq e \in \mathcal{H}$.

3.1. Complexity of the Ecpi *problem*

A chord in a path or in a cycle P is *odd* (resp. *even*) if it joins vertices at odd (resp. even) distance from each other in P. A graph is said *strongly chordal* [13] if it is chordal and, in addition every even cycle of length at least 6 has an odd chord [13, 12]. A path $P = (v_0, v_1, \dots, v_n)$ in a graph is *even-chorded* if it has no odd chord and neither v_0 nor v_n lies in a chord of P. Given two vertices u and v, the *even-chorded path interval operator* in a graph contains every vertices on every even-chorded path between u and v. A subset X of vertices of $V(G)$ is s-convex if it contain all the vertices in an even-chorded path between vertices of X. The class of strongly chordal graph is exactly the one for which the s-convexity is geometric [13].

As said in the introduction the strongly chordal graphs and the totally balanced hypergraphs are correlated to each other. In fact we have the following characterizations

- A graph G is strongly chordal if and only if $\mathcal{N}(G)$ is totally balanced. [12]
- A hypergraph is totally balanced if and only if its two-section is strongly chordal. [9]

The strongly chordal graphs are interesting also because some optimization problems, which are NP-complete in chordal graphs, become polynomially solvable in strongly chordal graphs [13]. We note the following

Remark 3. *In a bipartite graph a path is even-chorded if and only if it is a chordless path.*

and therefore

Remark 4. *In a bipartite graph the monophonic convexity and the s-convexity do coincide.*

In light of Remark 3 and 4 we have the following

Theorem 2. *The* Ecpi *problem is NP-complete.*

Proof: Clearly the Ecpi problem is in NP. By Remark 3 and 4 in a bipartite graph a path is even-chorded if and only if it is a chordless path. By Theorem 1, determining if a vertex in a bipartite graph belongs to an even-chorded path between two vertices is NP-complete. □

3.2. Complexity of the Spi *problem*

Let $\mathcal{H}$ be a hypergraph and u and v two vertices of $\mathcal{H}$. The *simple path interval operator* contains every vertex on every simple path between u and v. A subset X of vertices of $V(\mathcal{H})$ is s.p. convex if it contains all the vertices in a simple path between vertices of X. The class of totally balanced hypergraphs is exactly the one for which the simple path convexity is a convex geometry [13].

Given a connected bipartite graph G with at least two vertices and partition (X, Y) of $V(G)$ then the hypergraph $\mathcal{H}_{G(X)}$ associated to G with vertex set X is given by $\{N_G(y) : y \in Y\}$. We have the following

Lemma 5. *Let G be a connected bipartite graph with at least two vertices and bipartition (X, Y). Then a path $(x_0, y_1, x_1, \ldots, y_k, x_k)$, $x_0, x_k \in X$ is chordless in G if and only if $(x_0, N_G(y_1), x_1, \ldots, N_G(y_k), x_k)$ is a simple path of $\mathcal{H}_{G(X)}$.*

Proof: (*only if*) Suppose that $P = (x_0, y_1, x_1, \ldots, y_k, x_k)$ is chordless in G and let $P' = (x_0, N_G(y_1), x_1, \ldots, N_G(y_k), x_k)$. First we show that $N_G(y_i) \neq N_G(y_j)$ for all $i \neq j$. In fact suppose that there exist i and j, $i \neq j$, such that $N_G(y_i) = N_G(y_j)$. Then $x_{i-1} \in N_G(y_j)$ and $x_{i-1}y_j$ would be a chord of P. Therefore P' is a path of $\mathcal{H}_{G(X)}$. Suppose now that P' is not simple in $\mathcal{H}_{G(X)}$. Then there exists an edge $N_G(y_j)$ such that $N_G(y_j) \cap \{x_0, \ldots, x_k\} \neq \{x_{j-1}, x_j\}$. Let $x_i \in N_G(y_j) - \{x_{j-1}, x_j\}$, $i \in \{0, \ldots, k\}$. But then $x_i y_j$ is an edge of G and a chord of P, and a contradiction arises.
(*if*) Suppose that $P' = (x_0, N_G(y_1), x_1, \ldots, N_G(y_k), x_k)$ is a simple path in $\mathcal{H}_{G(X)}$. We claim that $P = (x_0, y_1, x_1, \ldots, y_k, x_k)$ is chordless. Suppose not and let $y_i x_j$ be a chord of P. Then $x_j \in N_G(y_i)$ and this implies that $N_G(y_i) \cap \{x_0, \ldots, x_k\} \neq \{x_{i-1}, x_i\}$, that is, P' is not simple. □

By Lemma 5, we have the following

Theorem 3. *The* Spi *problem is NP-complete.*

Proof: Clearly the Spi problem is in NP. We shall show a reduction from the independent set problem. Let G be a graph and k an integer and let G' be the bipartite graph as in Section 2. We add to G' a vertex s^0 and an edge $s^0 s^1$. Let (X, Y) be the partition of $V(G')$ such that X contains s^0, q^k and σ^1. Finally let $\mathcal{H}_{G'(X)}$ be the hypergraph with vertex set X associated to G'. By Lemma 5, $\mathcal{H}_{G'(X)}$ has a simple path between s^0 and σ^1 containing q^k if and only if G' has a chordless path between s^0 and σ^1 containing q^k. By Theorem 1, G' has a chordless path between s^0 and σ^1 containing q^k if and only if G has an independent set of size k. Since G' and $\mathcal{H}_{G'(X)}$ can be constructed in polynomial time from G this completes the proof. □

4. Conclusions

We showed that the Cp3v remains NP-complete in bipartite graphs, therefore strengthening previous results on the same problem. We showed also that this implies that the Spi and the Ecpi problems, are NP-complete. Interestingly, while the Spi problem is NP-complete, computing the simple path convex hull of a set of vertices of a hypergraph can be done in polynomial time [17].

Acknowledgment

The author wishes to thank the anonymous referee for his comments.

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