Exercise

March 24, 2023

Problem. Give a characterization of when a half-space $\mathscr{H}_1 = \{\mathbf{x} \in \mathbb{R}^n : c_1^T \mathbf{x} \leq b_1\}$ is contained in a half-space $\mathscr{H}_2 = \{\mathbf{x} \in \mathbb{R}^n : c_2^T \mathbf{x} \leq b_2\}$.

Solution. Note that as half-spaces, it holds that neither c_1 nor c_2 is equal to $\mathbf{0}$. First, as a case, assume that there does not exist $\alpha \in \mathbb{R}$ such that $c_1 = \alpha c_2$. We claim that $\mathscr{H}_1 \nsubseteq \mathscr{H}_2$.

Consider first the vector subspaces S_1 defined by the set of $\mathbf{x}$ such that $c_1^T \mathbf{x} = \mathbf{0}$. Then there exists $\mathbf{z} \in S_1$ and β such that $\beta c_1 + \mathbf{z} = c_2$. To see this, pick a basis $d_1, d_2, \dots, d_{n-1}$ of S_1. Then $d_1, \dots, d_{n-1}, c_1$ is a basis for $\mathbb{R}^n$ and thus there exist $\beta_1, \dots, \beta_{n-1}, \beta$ such that

$$c_2 = \beta_1 d_1 + \beta_2 d_2 + \cdots + \beta_{n-1} d_{n-1} + \beta c_1.$$

Let $\mathbf{z} = \beta_1 d_1 + \cdots + \beta_{n-1} d_{n-1}$. Note that by assumption, $\mathbf{z} \neq \mathbf{0}$.

Fix an element $\mathbf{y} \in \mathscr{H}_1$. It holds that

$$c_1^T (\mathbf{y} + \gamma \mathbf{z}) = c_1^T \mathbf{y} + \gamma c_1^T \mathbf{z} = c_1^T \mathbf{y} \leq b_1.$$

Thus $\mathbf{y} + \gamma \mathbf{z} \in \mathscr{H}_1$ for all $\gamma \in \mathbb{R}$.

If $c_2^T \mathbf{y} > b_2$, then $\mathbf{y} \notin \mathscr{H}_2$ and the claim holds. Assume instead that $c_2^T \mathbf{y} \leq b_2$. We have that

$$\begin{aligned} c_2^T (\mathbf{y} + \gamma \mathbf{z}) &= c_2^T \mathbf{y} + \gamma c_2^T \mathbf{z} \\ &= c_2^T \mathbf{y} + \gamma (\beta c_1 + \mathbf{z})^T z \\ &= c_2^T \mathbf{y} + \gamma \beta c_1^T \mathbf{z} + \gamma \mathbf{z}^T \mathbf{z} \\ &= c_2^T \mathbf{y} + \gamma \mathbf{z}^T \mathbf{z}. \end{aligned}$$

As $\mathbf{z} \neq \mathbf{0}$, we have that $\mathbf{z}^T \mathbf{z} > 0$ and thus for large values of γ, $c_2^T (\mathbf{y} + \gamma \mathbf{z}) > b_2$. Thus, $\mathbf{y} + \gamma \mathbf{z} \in \mathscr{H}_1$ but $\mathbf{y} + \gamma \mathbf{z} \notin \mathscr{H}_2$. We conclude that if $c_1 \neq \alpha c_2 \ \forall \alpha \in \mathbb{R}$, then $\mathscr{H}_1 \nsubseteq \mathscr{H}_2$.

Assume now that there exists $\alpha \in \mathbb{R}$ such that $c_1 = \alpha c_2$. Fix $\mathbf{y} \in \mathscr{H}_1$.

$$c_2^T \mathbf{y} = \frac{1}{\alpha} c_1^T \mathbf{y} \leq \frac{1}{\alpha} b_1.$$

Thus, $c_2^T \mathbf{y} \leq b_2$ if and only if $\frac{1}{\alpha} b_1 \leq b_2$. Note that for the "only if" to hold, we are using the property that there exist $\mathbf{y} \in \mathscr{H}_1$ such that $c_1^T \mathbf{y} = b_1$.

We conclude that $\mathscr{H}_1 \subseteq \mathscr{H}_2$ if and only if there exists $\alpha \in \mathbb{R}$ such that $c_1 = \alpha c_2$ and $\frac{1}{\alpha} b_1 \leq b_2$.