Stabilizer theorems for even cycle matroids

Bertrand Guenin
Dept. of Combinatorics and Optimization
University of Waterloo
200 University Avenue
Waterloo, ON, Canada

Irene Pivotto *
Dept. of Combinatorics and Optimization
University of Waterloo
200 University Avenue
Waterloo, ON, Canada

Paul Wollan
Dept. of Computer Science
University of Rome, *La Sapienza*
Via Salaria, 113
Rome, Italy

December 27, 2011

Abstract

A signed graph is a representation of an even cycle matroid M if the cycles of M correspond to the even cycles of that signed graph. Two long standing, open questions regarding even cycle matroids are the problem finding an excluded minor characterization and the problem of efficiently recognizing this class of matroids. Progress on these problems has been hampered by the fact that even cycle matroids can have an arbitrary number of pairwise inequivalent representations (two signed graph are equivalent if they are related by a sequence of Whitney-flips and signature exchanges). We show that we can bound the number of inequivalent representations of an even cycle matroid M (under some mild connectivity assumptions) if M contains any fixed size minor that is not a projection of a graphic matroid. For instance, any connected even cycle matroid which contains R_{10} as a minor has at most 6 inequivalent representations.

1 Introduction

We assume that the reader is familiar with the basics of matroid theory. See Oxley [6] for the definition of the terms used here. We will only consider binary matroids in this paper. Thus the reader should substitute the term "binary matroid" every time "matroid" appears in this text.

In this article, we will consider graphs with multiple edges and loops. Let G be a graph. For a set $X \subseteq E(G)$, we write $V_G(X)$ to refer to the set of vertices incident to an edge of X and $G[X]$ for

*Present address: Dept. of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby, BC, Canada; email:`ipivotto@sfu.ca`; phone:(+1)778 782 5754; fax:(+1)778 782 3332.

the subgraph with vertex set $V_G(X)$ and edge set X. A subset C of edges is a *cycle* if $G[C]$ is a graph where every vertex has even degree. An inclusion-wise minimal non-empty cycle is a *circuit*. Let G be a graph. We denote by $\mathrm{cycle}(G)$ the set of all cycles of G. The set $\mathrm{cycle}(G)$ is the set of cycles of the *graphic* matroid of G. We identify $\mathrm{cycle}(G)$ with that matroid. We say that G is a *representation* of that graphic matroid.

A *signed graph* is a pair (G,Σ) where G is a graph and $\Sigma \subseteq E(G)$. A subset $B \subseteq E(G)$ is *even* (resp. *odd*) if $|B \cap \Sigma|$ is even (resp. odd). In particular an edge e is odd if and only if $e \in \Sigma$. We say that Σ' is a *signature* of (G,Σ) if (G,Σ) and (G,Σ') have the same set of even cycles. Equivalently, it is straightforward to prove that Σ' is a signature if $\Sigma \triangle \Sigma'$ is a cut of G. In that case (G,Σ) and (G,Σ') are related by a *signature exchange*. Let (G,Σ) be a signed graph. We denote by $\mathrm{ecycle}(G,\Sigma)$ the set of all even cycles of (G,Σ). The set $\mathrm{ecycle}(G,\Sigma)$ is the set of cycles of a binary matroid known as the *even cycle matroid*. We identify $\mathrm{ecycle}(G,\Sigma)$ with that matroid. We say that (G,Σ) is a *representation* of that matroid. Observe that since $\mathrm{cycle}(G) = \mathrm{ecycle}(G,\emptyset)$, every graphic matroid is an even cycle matroid.

1.1 Representations of graphic matroids are nice

We will state a theorem that shows, for a graphic matroid, how to construct the set of all representations from a single representation. We require a number of definitions.

Let G be a graph and let $X \subseteq E(G)$. We write $\mathscr{B}_G(X)$ for $V_G(X) \cap V_G(\bar{X})$. [1] Suppose that $\mathscr{B}_G(X) = \{u_1,u_2\}$ for some $u_1,u_2 \in V(G)$. Let G' be the graph obtained by identifying vertices u_1,u_2 of $G[X]$ with vertices u_2,u_1 of $G[\bar{X}]$ respectively. Then G' is obtained from G by a *Whitney-flip* on X. We will also call Whitney-flip the operation consisting of identifying two vertices from distinct components, or the operation consisting of partitioning the graph into components each of which is a block of G. We define two graphs to be *equivalent* if one can be obtained from the other by a sequence of Whitney-flips (it is easy to verify that this does indeed define an equivalence relation).

In a seminal paper [13], Whitney proved the following.

Theorem 1. *A graphic matroid has a unique representation, up to equivalence.*

It follows in particular that, if a graphic matroid is 3-connected, then it has a unique representation.

1.2 Representations of even cycle matroids are naughty

The situation is considerably more complicated for even cycle matroids than for graphic matroids as we will illustrate in this section.

[1] $\bar{X} = E(G) - X$, where for any pair of sets A and B, $A - B = \{a \in A : a \notin B\}$. Throughout the paper we shall omit indices when there is no ambiguity. For instance we may write $\mathscr{B}(X)$ for $\mathscr{B}_G(X)$.

Suppose that (G_1,Σ_1) and (G_2,Σ_2) are signed graphs where G_1 and G_2 are equivalent and Σ_2 is is a signature of (G_1,Σ_1). Then we say that (G_1,Σ_1) and (G_2,Σ_2) are *equivalent*. Evidently, in that case $\mathrm{ecycle}(G_1,\Sigma_1) = \mathrm{ecycle}(G_2,\Sigma_2)$. Moreover, it can be easily checked that if G_1 and G_2 are equivalent graphs and $\mathrm{ecycle}(G_1,\Sigma_1) = \mathrm{ecycle}(G_2,\Sigma_2)$ for some signatures Σ_1 and Σ_2, then (G_1,Σ_1) and (G_2,Σ_2) are equivalent. Equivalent signed graphs do indeed define an equivalence relation. It follows that for any even cycle matroid N we can partition its representations into equivalence classes $\mathbb{F}_1,\dots,\mathbb{F}_k$. We will say that $\mathbb{F}_i$ ($i \in [k]$) is an *equivalence class of* N.

There is no direct analogue to Theorem 1 for even cycle matroids as the following result illustrates,

Remark 2. *For any integer k, there exists a even cycle matroid M with $|E(M)| \le 4k$ and 2^{k-1} equivalence classes.*

We now describe a general operation to construct the matroids given in the previous result.

Let G be a graph. Given $U \subseteq V(G)$, we denote by $\delta_G(U)$ the *cut* induced by U, that is $\delta_G(U) := \{(u,v) \in E(G) : u \in U, v \notin U\}$. We write $\delta_G(u)$ for $\delta_G(\{u\})$ for a vertex $u \in V(G)$. Given a graph G we denote by $\mathrm{loop}(G)$ the set of all loops of G. Let (G,Σ) be a signed graph. A vertex s is a *blocking vertex* of (G,Σ) if every odd circuit of (G,Σ) either contains the vertex s or is a loop. Similarly, a pair of vertices s,t is a *blocking pair* if every odd circuit of (G,Σ) either uses at least one of s and t or is a loop. Note that s is a blocking vertex (respectively s,t is a blocking pair) of (G,Σ) if and only if there exists a signature Σ' of (G,Σ) such that $\Sigma' \subseteq \delta(s) \cup \mathrm{loop}(G)$ (respectively $\Sigma' \subseteq \delta(s) \cup \delta(t) \cup \mathrm{loop}(G)$).

Consider a signed graph (G,Σ) and vertices $v_1, v_2 \in V(G)$, where $\Sigma \subseteq \delta_G(v_1) \cup \delta_G(v_2) \cup \mathrm{loop}(G)$. So v_1, v_2 is a blocking pair of (G,Σ). We can construct a signed graph (G',Σ) from (G,Σ) by replacing the endpoints x,y of every odd edge e with new endpoints x',y' as follows:

- if $x = v_1$ and $y = v_2$ then $x' = y'$ (i.e. e becomes a loop);
- if $x = y$ (i.e. e is a loop), then $x' = v_1$ and $y' = v_2$;
- if $x = v_1$ and $y \neq v_1, v_2$, then $x' = v_2$ and $y' = y$;
- if $x = v_2$ and $y \neq v_1, v_2$, then $x' = v_1$ and $y' = y$.

Then we say that (G',Σ) is obtained from (G,Σ) by a *Lovász-flip* on v_1, v_2. It is easy to show that Lovász-flips preserve even cycles [3, 4]. Using Lovász-flips we can construct inequivalent signed graphs representing the same even cycle matroid. An example is given in Figure 1. Each $G_1,\dots,G_4$ may stand for an arbitrary graph. As an example we chose G_1 to be the graph with edges $1,2,3,4,5,6$ given in the figure. The arrows indicate how each piece is flipped between the graph on the left and the graph on the right. The odd edges, in both signed graphs, are $1,2,3$. Note that, for every $i \in [4]$, the two vertices in $V_{G_i} \cap V_{G_{i+1}}$ form a blocking pair. It is possible to obtain the signed graph on the right from the signed graph on the left by signature exchanges and Lovász-flips on each of these blocking pairs. This construction generalizes to any number of graphs $G_1,\dots,G_k$ and using Lovász-flips and signature

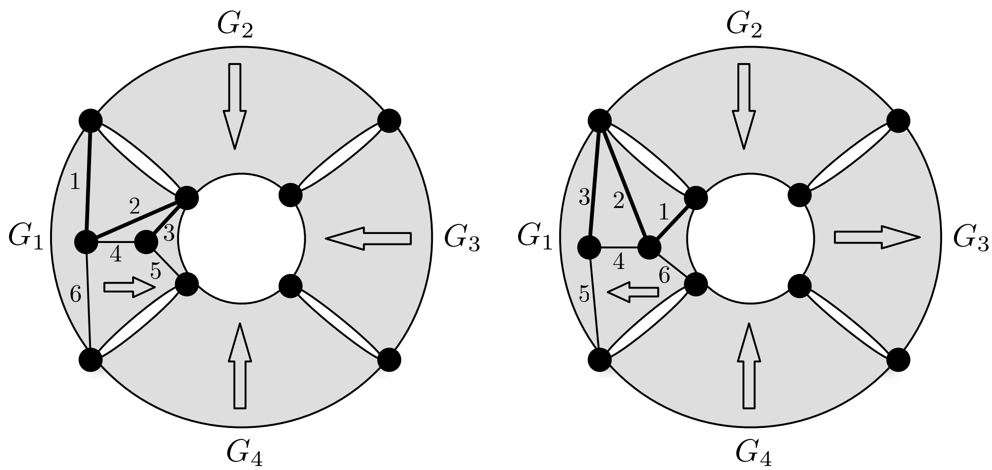

Figure 1: Inequivalent signed graphs representing the same matroid.

exchanges we can flip any subset of these k graphs. In particular, it is easy to construct matroids M as in Remark 2 (for each $i \in [k]$ let G_i be a complete bipartite graph with 2 vertices on each sides).

As Remark 2 shows, if a signed graph (G,Σ) has blocking pairs then $\text{ecycle}(G,\Sigma)$ may have many inequivalent representations. On the other hand, if a signed graph has a blocking pair, then it cannot have three, pairwise vertex disjoint, odd circuits. Thus one may wonder if having three, pairwise vertex disjoint, odd circuits, forces the representation to be unique, up to equivalence. Slilaty [10] proved that the analogue of this statement holds for *signed-graphic* matroids. Alas, no similar result holds for even cycle matroids, as the following remark indicates. It shows that blocking pairs are not the only reason for having inequivalent representations.

Remark 3. *For every integer k, there exists a signed graph (G,Σ) with the property that:*

(1) every signed graph equivalent to (G,Σ) has k, pairwise vertex disjoint, odd circuits, and

(2) $\text{ecycle}(G,\Sigma)$ has at least two inequivalent representations.

We postpone the proof of this remark until Section 4.2.2.

1.3 Main results

Given a matroid M and disjoint subsets $I,J \subseteq E(M)$, the matroid $M \setminus I/J$ denotes the minor of M obtained by deleting the elements in I and contracting the elements in J. We define minor operations on signed graphs as follows. Let (G,Σ) be a signed graph and let $e \in E(G)$. Then $(G,\Sigma)\setminus e$ is defined as $(G\setminus e, \Sigma - \{e\})$. [2] We define $(G,\Sigma)/e$ as $(G\setminus e, \emptyset)$ if e is an odd loop of (G,Σ) and as $(G\setminus e,\Sigma)$ if e

[2] Given a graph G and $e \in E(G)$, $G\setminus e$ is the graph obtained by deleting e whereas G/e is the graph obtained by contracting e.

is an even loop of (G,Σ); otherwise $(G,\Sigma)/e$ is equal to $(G/e,\Sigma')$, where Σ' is any signature of (G,Σ) which does not contain e. Observe that (see [5] for instance),

Remark 4. $\text{ecycle}(G,\Sigma)\setminus I/J = \text{ecycle}\big((G,\Sigma)\setminus I/J\big)$.

In particular, this implies that being an even cycle matroid is a minor closed property.

1.3.1 Non-degenerate minors

We say that an even cycle matroid is *degenerate* if any of its representation has a blocking pair. If a signed graph has a blocking pair, then so does every minor. It follows from Remark 4 that being degenerate is a minor closed property. If an even cycle matroid N is graphic, then it has a representation $(G,\emptyset)$ as an even cycle matroid, and trivially, any pair of vertices of G is a blocking pair. Hence, graphic matroids are degenerate. An example of an even cycle matroid that is non-degenerate is given by the matroid R_{10} (introduced in [9]). R_{10} has six representations as an even cycle matroid, all isomorphic to the signed graph $\big(K_5,E(K_5)\big)$. (How to find these representations is explained in [5].) This signed graph does not have a blocking pair, as the removal of any two vertices leaves an odd triangle.

We are now ready to present the first main result of the paper,

Theorem 5. *Let M be a 3-connected even cycle matroid which contains as a minor a non-degenerate 3-connected matroid N. Then the number of equivalence classes of M is at most twice the number of equivalence classes of N.*

This result implies, in particular, that every 3-connected even cycle matroid containing R_{10} as a minor has, up to equivalence, at most 12 representations. We will strengthen this result in Section 1.3.2.

We will show that degenerate matroids are "close" to being graphic matroids. We require a number of definitions to formalize this notion.

Consider a graph H with a vertex v and $\alpha \subseteq \delta_H(v)\cup\text{loop}(H)$. We say that G is obtained from H by *splitting* v into v_1 and v_2 according to α if $V(G)=V(H)-\{v\}\cup\{v_1,v_2\}$ and for every $e=(u,w)\in E(H)$:

- if $e\notin\delta_H(v)\cup\text{loop}(H)$, then $e=(u,w)$ in G;
- if $e\in\text{loop}(H)\cap\alpha$, then $e=(v_1,v_2)$ in G;
- if $e\in\delta_H(v)\cap\alpha$ and $u\neq v, w=v$, then $e=(u,v_1)$ in G;
- if $e\in\delta_H(v)-\alpha$ and $u\neq v, w=v$, then $e=(u,v_2)$ in G.

Let N and M be matroids where $E(N)=E(M)$. Then N is a *lift* of M if, for some matroid M' where $E(M')=E(M)\cup\{\Omega\}$, $M=M'/\Omega$ and $N=M'\setminus\Omega$. If N is a lift of M then M is a *projection* of N. Lifts and projections were introduced in [2]. Every even cycle matroid M is a lift of a graphic matroid;

indeed, for any representation (G,Σ) of M we may construct (G',Σ') by adding an odd loop Ω. Then $\mathrm{ecycle}(G',\Sigma')/\Omega$ is a graphic matroid. The following result shows that degenerate even cycle matroids are projections of graphic matroids.

Remark 6. *Let (H,Γ) be a signed graph.*

(1) If (H,Γ) has a blocking vertex, then $\mathrm{ecycle}(H,\Gamma)$ *is a graphic matroid.*

(2) If (H,Γ) has a blocking pair, then $\mathrm{ecycle}(H,\Gamma)$ *is a projection of a graphic matroid.*

Proof. **(1)** Suppose that $\Gamma \subseteq \delta_H(s) \cup \mathrm{loop}(H)$ for some vertex s of H. Let G be obtained from H by splitting s according to Γ. Then $\mathrm{cycle}(G) = \mathrm{ecycle}(H,\Gamma)$. **(2)** Suppose that $\Gamma \subseteq \delta_H(s) \cup \delta_H(t) \cup \mathrm{loop}(H)$ for a pair of vertices s,t of H. Let G be obtained from H by splitting s into s_1 and s_2 according to $\delta_H(s) \cap \Gamma$ and by adding an edge $\Omega = (s_1,s_2)$. Let $M' = \mathrm{ecycle}(G,\Gamma)$. Then by construction $(G,\Gamma)/\Omega = (H,\Gamma)$, hence $M'/\Omega = M$. Moreover, by (1), $\mathrm{ecycle}(G,\Gamma) \setminus \Omega = M' \setminus \Omega$ is a graphic matroid, as t is a blocking vertex of $(G,\Gamma) \setminus \Omega$. $\square$

1.3.2 Substantial minors

Consider a signed graph (G,Σ) and suppose that there exists a partition $\mathscr{C}_1, \mathscr{C}_2$ of the odd circuits of (G,Σ) and graphs G_1 and G_2 equivalent to G such that, for $i = 1,2$, some $v_i \in V(G_i)$ intersects all circuits in $\mathscr{C}_i$. Then we call the pair (G_1,v_1) and (G_2,v_2) an *intercepting pair* for (G,Σ). In particular, if (G,Σ) has a blocking pair v_1,v_2, then (G,v_1), (G,v_2) is an intercepting pair for (G,Σ). An even cycle matroid is *substantial* if none of its representations has an intercepting pair. Hence, if an even cycle matroid is degenerate it is not substantial. In particular, substantial matroids are not graphic. We will see (Remark 13) that not being substantial is also a minor closed property. If, for every representation (G,Σ) of an even cycle matroid M, the graph G is 3-connected and (G,Σ) has no blocking pair, then M is substantial. As all 6 representations of R_{10} are isomorphic to $\big(K_5, E(K_5)\big)$, R_{10} is substantial.

We are now ready to present the second main result of the paper,

Theorem 7. *Let M be a connected even cycle matroid which contains as a minor a connected matroid N that is substantial. Then the number of equivalence classes of M is at most the number of equivalence classes of N.*

This result implies, in particular, that every connected even cycle matroid containing R_{10} as a minor has, up to equivalence, at most 6 representations.

1.4 Related results and motivation

Even cycle matroids are a natural class of matroids to study as they are the smallest minor closed class of binary matroids which contains all single element co-extensions of graphic matroids. Robertson and Seymour [7] proved that for every infinite set of graphs one of its members is isomorphic to a minor of another. Gerards, Geelen, and Whittle announced that an analogous result holds for binary matroids. Hence, any minor closed class of binary matroids can be characterized by a finite set of excluded minors. In particular this is the case for even cycle matroids. Tutte [12] gave an explicit description of the excluded minors for the class of graphic matroids. He also gave a polynomial time algorithm to check if a binary matroid (given by its $0,1$ matrix representation) is graphic [11].

No explicit description of the excluded minors is known for even cycle matroids and we do not know how to recognize efficiently whether a given binary matroid is an even cycle matroid. The difficulty for both problems lies with the fact that we do not have a sufficient understanding of the representations of even cycle matroids. Theorems 5 and 7 are a first step towards a better understanding of this problem. Eventually, we wish to extend the aforementioned theorems so as to have a compact description of the representations of arbitrary even cycle matroids. We believe that there exists a constant k such that every even cycle matroid with more than k inequivalent representations is constructed in a way analogous to that of the example in Section 1.2. The problem of describing the pairwise relationship between any two representations of an even cycle matroid is discussed in [4].

1.5 Organization of the paper

Section 2 introduces generalizations of Theorems 5 and 7. An outline of the proofs of these theorems is then presented leaving out two key lemmas, namely 15 and 16. Lemma 15 is proved in Section 3. Section 4 we prove a characterization of class of inequivalent representations of even cycle matroids. This is required for the proof of Lemma 16 which is then given in Section 5.

2 The proofs (modulo the exclusion of several lemmas)

If N is a minor of a matroid M then M is a *major* of N. Consider an even cycle matroid M with a representation (G,Σ). Let I and J be disjoint subsets of $E(M)$ and let $N := M \setminus I/J$. Let $(H,\Gamma) := (G,\Sigma) \setminus I/J$. It follows from Remark 4 that (H,Γ) is a representation of N. We say that (G,Σ) is an *extension* to M of the representation (H,Γ) of N, or alternatively that (H,Γ) *extends* to M.

The following result implies Theorem 5.

Theorem 8. *Let N be a 3-connected non-degenerate even cycle matroid. Let M be a 3-connected major of N. For every equivalence class $\mathbb{F}$ of N, the set of extensions of $\mathbb{F}$ to M is the union of at most two equivalence classes.*

The following result implies Theorem 7.

Theorem 9. *Let N be a connected substantial even cycle matroid. Let M be a connected major of N. For every equivalence class $\mathbb{F}$ of N, the set of extensions of $\mathbb{F}$ to M is contained in one equivalence class.*

The proofs of Theorems 8 and 9 are constructive. Thus, given a description of the inequivalent representations of N, it is possible to construct the set all inequivalent representations of M.

2.1 Definitions

First an easy observation,

Remark 10. *If G_1 and G_2 are equivalent graphs, then $G_1 \setminus I/J$ and $G_2 \setminus I/J$ are equivalent.*

Proof. Since G_1 and G_2 are equivalent, $\text{cycle}(G_1) = \text{cycle}(G_2)$. Hence, $\text{cycle}(G_1) \setminus I/J = \text{cycle}(G_2) \setminus I/J$. As the minor operations on graphs and matroid commute, we have, $\text{cycle}(G_1 \setminus I/J) = \text{cycle}(G_2 \setminus I/J)$. The result now follows from Theorem 1. □

Consider a matroid M and let $N := M \setminus I/J$ be a minor of M. If $J = \emptyset$ and $|I| = 1$ then M is a *column major* of N. If $I = \emptyset$ and $|J| = 1$ then M is a *row major* of N. A set $\mathbb{F}$ of representations of an even cycle matroid is *closed under equivalence* if, for every $(H,\Gamma) \in \mathbb{F}$ and (H',Γ') equivalent to (H,Γ), we have that $(H',\Gamma') \in \mathbb{F}$.

Remark 11. *Let $\mathbb{F}$ be a set of representations of an even cycle matroid N and let M be a major of N. If $\mathbb{F}$ is closed under equivalence, then so is the set $\mathbb{F}'$ of extensions of $\mathbb{F}$ to M.*

Proof. Let $(G_1,\Sigma_1) \in \mathbb{F}'$ and let (G_2,Σ_2) be equivalent to (G_1,Σ_1). We have $N = M \setminus I/J$, for some $I,J \subseteq E(M)$. By definition of $\mathbb{F}'$ there exists $(H_1,\Gamma_1) \in \mathbb{F}$ where $(H_1,\Gamma_1) = (G_1,\Sigma_1) \setminus I/J$. Let $(H_2,\Gamma_2) := (G_2,\Sigma_2) \setminus I/J$. Remark 10 implies that (H_1,Γ_1) and (H_2,Γ_2) are equivalent. As $\mathbb{F}$ is closed under equivalence, $(H_2,\Gamma_2) \in \mathbb{F}$. Since, $\mathbb{F}'$ is the set of extension of $\mathbb{F}$ to M, $(G_2,\Sigma_2) \in \mathbb{F}'$. □

Let $\mathbb{F}$ be an equivalence class of an even cycle matroid N that is not graphic. We say that $\mathbb{F}$ is *row stable* (resp. *column stable*) if for all row (resp. column) majors M of N, where

- M has no loop, and no co-loop, and
- M is not graphic,

the set of extensions of $\mathbb{F}$ to M is an equivalence class.

2.2 A sketch of the proof of Theorem 9

We postpone the proof of the following result until Section 2.4.

Lemma 12. *Every equivalence class of an even cycle matroid is column stable.*

The following implies that if a matroid is not substantial then neither are any of its minors.

Remark 13. *If (G,Σ) has an intercepting pair, then so does every minor (H,Γ) of (G,Σ).*

A signed graph (G,Σ) is *bipartite* if all cycles are even. We require the following observation,

Remark 14. *Suppose (G,Σ) has an intercepting pair (G_1,v_1) and (G_2,v_2). Then there exists for $i=1,2$, $\alpha_i \subseteq \delta_{G_i}(v_i) \cup \mathrm{loop}(G)$, such that $\alpha_1 \triangle \alpha_2$ is a signature of (G,Σ).*

Proof. Every odd circuit of (G,Σ) is a circuit of G_1 using v_1 or a circuit of G_2 using v_2. It follows that $(G,\Sigma) \setminus \left[\delta_{G_1}(v_1) \cup \delta_{G_2}(v_2) \cup \mathrm{loop}(G_1) \cup \mathrm{loop}(G_2)\right]$ is bipartite. Since $\mathrm{loop}(G_1) = \mathrm{loop}(G_2) = \mathrm{loop}(G)$, there is a signature of (G,Σ) contained in $\delta_{G_1}(v_1) \cup \delta_{G_2}(v_2) \cup \mathrm{loop}(G)$ and the result follows. $\square$

Proof of Remark 13. Suppose (G,Σ) has an intercepting pair $(G_1,v_1),(G_2,v_2)$. Let $\alpha_1 \triangle \alpha_2$ be the signature of (G,Σ) given by Remark 14. We have $(H,\Gamma) = (G,\Sigma) \setminus I/J$ for some $I,J \subseteq E(G)$. For $i=1,2$, let $(H_i,\beta_i) := (G_i,\alpha_i) \setminus I/J$. Remark 10 implies that H_1 and H_2 are equivalent. For $i=1,2$, v_i is a blocking vertex of (G_i,α_i). Hence, there is a blocking vertex w_i of (H_i,β_i) and we may assume that $\beta_i \subseteq \delta_{H_i}(w_i) \cup \mathrm{loop}(H_i)$. It follows from the definition of signed minor that, for some cut B_i of G_i, $\beta_i = (\alpha_i \triangle B_i) - I$ and $(\alpha_i \triangle B_i) \cap J = \emptyset$. Since G, G_1, G_2 are equivalent, B_1, B_2 are cuts of G, hence $\alpha_1 \triangle \alpha_2 \triangle B_1 \triangle B_2$ is a signature of (G,Σ). Hence, $\beta_1 \triangle \beta_2$ is a signature of (H,Γ). In particular, every odd circuit of (H,Γ) is a circuit of H_i using vertex w_i for some $i \in [2]$, i.e. $(H_1,w_1),(H_2,w_2)$ is an intercepting pair of (H,Γ). $\square$

We say that an equivalence class $\mathbb{F}$ has no intercepting pair if none of the signed graphs in $\mathbb{F}$ have an intercepting pair. Note, that we could replace "none" by "any" in the previous definition, as by definition, if a signed graph has an intercepting pair, then so does every equivalent signed graph.

We postpone the proof of the following result until Section 3.

Lemma 15. *Equivalence classes without intercepting pairs are row stable.*

***Proof of Theorem 9**.* Let N be a connected even cycle matroid, where none of the representations of N has an intercepting pair. Let M be a connected major of N. It follows (by [1, 8]) that there exists a sequence of connected matroids $N_1,\dots,N_k$, where $N=N_1$, $M=N_k$ and, for $i \in [k-1]$, N_{i+1} is a row or column major of N_i. In particular, N_i has no loops or co-loops, for every $i \in [k]$. Since N_1 is substantial, it is not graphic, hence neither are $N_2,\dots,N_k$. Let $\mathbb{F}$ be an equivalence class of N that extends to M and, for every $j \in [k]$, define $\mathbb{F}_j$ to be the set of extensions of $\mathbb{F}$ to N_j. It suffices to show that, for all $j \in [k]$, $\mathbb{F}_j$ is an equivalence class. Let us proceed by induction. As $N_1 = N$, the result holds for $j=1$. Suppose that the result holds for $j \in [k-1]$. By Remark 13, $\mathbb{F}_j$ does not have an intercepting pair. Therefore, by Lemma 12 and Lemma 15, $\mathbb{F}_j$ is column and row stable, respectively. It follows that $\mathbb{F}_{j+1}$ is an equivalence class. $\square$

2.3 A sketch of the proof of Theorem 8

We say that an equivalence class $\mathbb{F}$ has no blocking pair if none of the signed graphs in $\mathbb{F}$ have a blocking pair. We postpone the proof of the following result until Section 5.

Lemma 16. *Let N be an even cycle matroid and let $\mathbb{F}$ be an equivalence class of N with no blocking pair. Let M be a row major of N with no loops or co-loops. Suppose that N and M are 3-connected and suppose that the set $\mathbb{F}'$ of extensions of $\mathbb{F}$ to M is non-empty. Then $\mathbb{F}'$ is either an equivalence class or the union of two equivalence classes $\mathbb{F}_1$ and $\mathbb{F}_2$ without intercepting pairs.*

Proof of Theorem 8. Let N,M be as in the statement of the theorem. Since N is non-degenerate, it is non-graphic. It follows (by [9], as N is not the the graphic matroid of a wheel) that there is a sequence of 3-connected matroids $N_1,\dots,N_k$, where $N = N_1$, $M = N_k$ and, for every $i \in [k-1]$, N_{i+1} is a row or column major of N_i. In particular, N_i has no loops or co-loops for any $i \in [k]$. Since $N = N_1$ is not graphic, neither are $N_2,\dots,N_k$. Let $\mathbb{F}$ be an equivalence class of N that extends to M. For every $j \in [k]$, define $\mathbb{F}_j$ to be the set of extensions of $\mathbb{F}$ to N_j. It suffices to show that, for all $j \in [k]$, $\mathbb{F}_j$ is either

(a) an equivalence class, or

(b) the union of two equivalence classes without intercepting pairs.

Let us proceed by induction. As $N_1 = N$, the result holds for $j = 1$. Suppose that the result holds for $j \in [k-1]$. Consider the case where N_{j+1} is a column major of N_j. If (a) holds for $\mathbb{F}_j$, then Lemma 12 implies that (a) holds for $\mathbb{F}_{j+1}$. If (b) holds for $\mathbb{F}_j$, then Lemma 12 and Remark 13 imply that either (a) or (b) holds for $\mathbb{F}_{j+1}$. Consider the case where N_{j+1} is a row major of N_j. If (a) holds for $\mathbb{F}_j$, then Lemma 16 implies that either (a) or (b) holds. If (b) holds for $\mathbb{F}_j$, then Lemma 15 implies that either of (a) or (b) holds for $\mathbb{F}_{j+1}$. □

2.4 Proof of Lemma 12

The next result, proved in [4], is an easy consequence of Theorem 1.

Remark 17. *Suppose that* $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$. *If an odd cycle of (G_1,Σ_1) is a cycle of G_2, then G_1 and G_2 are equivalent.*

We say that two signed graphs (G_1,Σ_1) and (G_2,Σ_2) are *siblings* if $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$ and graphs G_1 and G_2 are not equivalent.

Lemma 18. *Let (G_1,Σ_1) and (G_2,Σ_2) be siblings and let $\Omega \in E(G_1)$. For $i = 1,2$, let $(H_i,\Gamma_i) := (G_i,\Sigma_i) \setminus \Omega$. Suppose that (H_1,Γ_1) and (H_2,Γ_2) are equivalent. Then, for $i = 1,2$, Ω is either a bridge of G_i or a signature of (G_i,Σ_i). In particular, Ω is a co-loop of* $\text{ecycle}(G_1,\Sigma_1)$.

Proof. We prove the statement for $i = 1$. Remark 17 implies that no odd cycle of (G_1,Σ_1) is a cycle of G_2. Since H_1 and H_2 are equivalent, $\text{cycle}(H_1) = \text{cycle}(H_2)$. It follows that all odd cycles of (G_1,Σ_1) use Ω. Hence, after possibly a signature exchange, $\Sigma_1 \subseteq \{\Omega\}$. Similarly, we may assume that $\Sigma_2 \subseteq \{\Omega\}$. If Ω is a bridge of G_1, we are done. Suppose otherwise. If $\Sigma_1 = \emptyset$, then there exists an even cycle C of (G_1,Σ_1) using Ω; hence Ω is not a bridge of G_2 and $\Sigma_2 \neq \{\Omega\}$. But then $\Sigma_1 = \Sigma_2 = \emptyset$ and $\text{cycle}(G_1) = \text{cycle}(G_2)$. It follows by Theorem 1 that G_1 and G_2 are equivalent, a contradiction. □

Proof of Lemma 12. Let $\mathbb{F}$ be an equivalence class of an even cycle matroid N. Let M be a column extension of N, i.e. for some $\Omega \in E(M)$, $N = M \setminus \Omega$. Let $\mathbb{F}'$ be the set of all extensions of $\mathbb{F}$ to M. Assume M has no co-loops. We need to show that $\mathbb{F}'$ is an equivalence class. For otherwise there exists siblings $(G_1,\Sigma_1),(G_2,\Sigma_2) \in \mathbb{F}'$. For $i = 1,2$, let $(H_i,\Gamma_i) := (G_i,\Sigma_i) \setminus \Omega$. Then $(H_1,\Gamma_1),(H_2,\Gamma_2) \in \mathbb{F}$. In particular, (H_1,Γ_1) and (H_2,Γ_2) are equivalent. Hence, by Lemma 18, Ω is a co-loop of $M = \text{ecycle}(G_1,\Sigma_1)$, a contradiction. □

It remains to prove Lemma 15 and 16. Lemma 15 (resp. 16) is proved in Section 3 (resp. 5).

3 Row extensions and intercepting pairs

Before we proceed with the proof of Lemma 15 we establish some preliminaries in Sections 3.1 and 3.2.

3.1 Even cut matroids

Given a graph G, we denote by $\text{cut}(G)$ the set of all cuts of G. Since the cuts of G correspond to the cycles of the *co-graphic matroid* of G, we identify $\text{cut}(G)$ with that matroid.

A *graft* is a pair (G,T) where G is a graph, $T \subseteq V(G)$ and $|T|$ is even. A cut $\delta(U)$ is *even* (respectively *odd*) if $|T \cap U|$ is even (respectively odd). We denote by $\text{ecut}(G,T)$ the set of all even cuts of (G,T). The set $\text{ecut}(G,T)$ is the set of cycles of a binary matroid known as the *even cut matroid*. We identify $\text{ecut}(G,T)$ with that matroid. Given a graph H, we denote by $V_{odd}(H)$ the set of vertices of H of odd degree.

We will make repeated use of the following result (which was proved in [4]).

Theorem 19. *Let G_1 and G_2 be inequivalent graphs.*

(1) Suppose that there exists a pair $\Sigma_1,\Sigma_2 \subseteq E(G_1)$ such that $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$*. For $i = 1,2$, if (G_i,Σ_i) is bipartite define $C_i := \emptyset$; otherwise let C_i be an odd cycle of (G_i,Σ_i). Let $T_i := V_{odd}(G_i[C_{3-i}])$. Then* $\text{ecut}(G_1,T_1) = \text{ecut}(G_2,T_2)$.

(2) Suppose that there exists a pair $T_1 \subseteq V(G_1), T_2 \subseteq V(G_2)$ (where $|T_1|$ and $|T_2|$ are even) such that $\text{ecut}(G_1,T_1) = \text{ecut}(G_2,T_2)$*. For $i = 1,2$, if $T_i = \emptyset$ let $\Sigma_{3-i} = \emptyset$; otherwise let $t_i \in T_i$ and $\Sigma_{3-i} := \delta_{G_i}(t_i)$. Then* $\text{ecycle}\ (G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$.

Moreover, if they exist, the pairs Σ_1, Σ_2 and T_1, T_2 are unique (up to signature exchange).

3.2 Split siblings

Consider a pair of equivalent graphs H_1 and H_2. Suppose that, for $i = 1,2$, we have $\alpha_i \subseteq \delta_{H_i}(v_i) \cup \text{loop}(H_i)$ for some $v_i \in V(H_i)$. Then, for $i = 1,2$, let G_i be obtained from H_i by splitting v_i into v_i^- and v_i^+ according to α_i and let $T_i := \{v_i^-, v_i^+\}$. As H_1 and H_2 are equivalent, $\text{cycle}(H_1) = \text{cycle}(H_2)$. Since

$\mathrm{cut}(H_1)$ and $\mathrm{cut}(H_2)$ are the duals of $\mathrm{cycle}(H_1)$ and $\mathrm{cycle}(H_2)$ we have that $\mathrm{cut}(H_1) = \mathrm{cut}(H_2)$. For $i = 1,2$, if $\delta_{G_i}(U)$ is an even cut of (G_i, T_i) then $T_i \subseteq U$ or $T_i \subseteq \bar{U}$ (because $|T_i| = 2$). Hence,

$$\mathrm{ecut}(G_1, T_1) = \mathrm{cut}(H_1) = \mathrm{cut}(H_2) = \mathrm{ecut}(G_2, T_2).$$

If G_1 is not equivalent to G_2 then Theorem 19 implies that there is a unique pair of signatures Σ_1 and Σ_2 (up to signature exchanges) such that $\mathrm{ecycle}(G_1, \Sigma_1) = \mathrm{ecycle}(G_2, \Sigma_2)$. We say, in that case, that (G_1, Σ_1) and (G_2, Σ_2) are *split siblings*. Observe that, in the previous definition, if Ω is a loop of H_1, H_2 contained in $\alpha_1 \cap \alpha_2$, then for $i = 1,2$, Ω has endpoints T_i in G_i. We will refer to split siblings with such an edge Ω as Ω-*split siblings*.

In light of the previous discussion, we say that a tuple $\mathbb{T} = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2)$, where H_1, H_2 are 2-connected (up to loops), is a *split-template* if the following conditions hold:

(a) H_1 and H_2 are equivalent graphs;

(b) for $i = 1,2$, $v_i \in V(H_i)$;

(c) for $i = 1,2$, $\alpha_i \subseteq \delta_{H_i}(v_i) \cup \mathrm{loop}(H_i)$.

We say that the split siblings (G_1, Σ_1) and (G_2, Σ_2) defined in the previous paragraph *arise* from the split-template $\mathbb{T}$.

Remark 20. *Let $\mathbb{T} = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2)$ be a split-template and let (G_1, Σ_1) and (G_2, Σ_2) be split siblings that arise from $\mathbb{T}$. Then, up to signature exchange, we have $\Sigma_1 = \Sigma_2 = \alpha_1 \triangle \alpha_2$.*

Proof. For $i = 1,2$, vertex v_i of H_i gets split into vertices v_i^- and v_i^+ of G_i. Let Ω denote the set of loops of both H_1 and H_2 that are in $\alpha_1 \cap \alpha_2$. Note, that, for $i = 1,2$, all edges in Ω will have both ends in T_i. By construction, $\alpha_i \cup \Omega = \delta_{G_i}(v_i^-)$, for $i = 1,2$. As $v_1^- \in T_1$, Theorem 19 implies that $\alpha_1 \cup \Omega$ is a signature of (G_2, Σ_2). As $\alpha_2 \cup \Omega$ is a cut of G_2, $\alpha_1 \triangle \alpha_2$ is a signature of (G_2, Σ_2). By symmetry, $\alpha_1 \triangle \alpha_2$ is also a signature of (G_1, Σ_1). Finally, by Theorem 19, Σ_1 and Σ_2 are unique up to signature exchanges. $\square$

3.3 Proof Lemma 15

The following easy observation is the analogue to Remark 17 for the case of even cut matroids (see [4]).

Remark 21. *Suppose that* $\mathrm{ecut}(G_1, T_1) = \mathrm{ecut}(G_2, T_2)$. *If any odd cut of (G_1, T_1) is a cut of G_2, then G_1 and G_2 are equivalent.*

Let (G, T) be a graft and let $e \in E(G)$. Then $(G, T) \setminus e$ is defined as $(G \setminus e, T')$, where $T' = \emptyset$ if e is an odd bridge of (G, T) and $T' = T$ otherwise. $(G, T)/e$ is equal to $(G/e, T')$, where T' is defined as follows. Let u, v be the ends of e in G and let w be the vertex obtained by contracting e. If $x \neq w$, then $x \in T'$ if and only if $x \in T$; $w \in T'$ if and only if $|\{u, v\} \cap T| = 1$. Observe that (see [5] for instance),

Remark 22. $\operatorname{ecut}(G,\Sigma)\setminus I/J = \operatorname{ecut}\big((G,\Sigma)/I\setminus J\big)$.

In particular, this implies that being an even cut matroid is a minor closed property.

The following result is the analogue to Lemma 18 for even cut matroids.

Lemma 23. *Suppose that G_1 and G_2 are not equivalent and* $\operatorname{ecut}(G_1,T_1) = \operatorname{ecut}(G_2,T_2)$. *Let $\Omega \in E(G_1)$. For $i = 1,2$, let $(H_i,R_i) := (G_i,T_i)/\Omega$. Suppose that H_1 and H_2 are equivalent. Then, for $i = 1,2$, either Ω is a loop of G_i or $|T_i| = 2$ and T_i are the ends of Ω in G_i. In particular, Ω is a co-loop of* $\operatorname{ecut}(G_1,T_1)$.

Proof. For $i = 1,2$, denote by v_i and w_i the endpoints of edge Ω in G_i. We prove the statement for $i = 1$. Remark 21 implies that no odd cut of (G_1,T_1) is a cut of G_2. Since H_1 and H_2 are equivalent, $\operatorname{cut}(H_1) = \operatorname{cut}(H_2)$. It follows that all odd cuts of (G_1,T_1) use Ω. Hence, $T_1 \subseteq \{v_1,w_1\}$. Similarly, we may assume that $T_2 \subseteq \{v_2,w_2\}$. If Ω is a loop of G_1, we are done. Suppose otherwise. If $T_1 = \emptyset$, then there exists an even cut B of (G_1,T_1) using Ω; hence Ω is not a loop of G_2 and $T_2 \neq \{v_2,w_2\}$. But then $T_1 = T_2 = \emptyset$ and $\operatorname{cut}(G_1) = \operatorname{cut}(G_2)$. Hence $\operatorname{cycle}(G_1) = \operatorname{cycle}(G_2)$ and it follows by Theorem 1 that G_1 and G_2 are equivalent, a contradiction. We conclude that $T_1 = \{v_1,w_1\}$. ☐

Lemma 24. *Let N be a non-graphic even cycle matroid and let $\mathbb{F}$ be an equivalence class of N. Let M be a row major of N with no loops or co-loops. Let Ω denote the unique element in $E(M) - E(N)$. Suppose that the set $\mathbb{F}'$ of extensions of $\mathbb{F}$ to M is non-empty. Then $\mathbb{F}'$ is either an equivalence class or the union of two equivalence classes $\mathbb{F}_1$ and $\mathbb{F}_2$ and any $(G_1,\Sigma_1) \in \mathbb{F}_1$ and $(G_2,\Sigma_2) \in \mathbb{F}_2$ are Ω-split siblings.*

Proof. We may assume that $\mathbb{F}'$ is not an equivalence class. Hence, there exist siblings $(G_1,\Sigma_1),(G_2,\Sigma_2) \in \mathbb{F}'$. For $i = 1,2$ let $(H_i,\Gamma_i) := (G_i,\Sigma_i)/\Omega$. By definition of $\mathbb{F}'$, $(H_1,\Gamma_1),(H_2,\Gamma_2) \in \mathbb{F}$. In particular, H_1 and H_2 are equivalent. Since G_1 and G_2 are not equivalent, Theorem 19 implies that there exists a unique pair $T_1 \subseteq V(G_1)$, $T_2 \subseteq V(G_2)$ such that $\operatorname{ecut}(G_1,T_1) = \operatorname{ecut}(G_2,T_2)$. For $i = 1,2$, we have $(H_i,R_i) = (G_i,T_i)/\Omega$ for some $R_1 \subseteq V(H_1)$ and $R_2 \subseteq V(H_2)$. Lemma 23 implies that, for $i = 1,2$, either Ω is a loop of G_i or T_i are the ends of Ω in G_i. If the latter case occurs for both $i = 1,2$, then we are done as (G_1,Σ_1) and (G_2,Σ_2) are Ω-split siblings (by Theorem 19, the pair Σ_1,Σ_2 is uniquely determined). Now suppose that Ω is a loop of G_i, for $i = 1$ or $i = 2$. Then every cut of G_i is a cut of H_i, hence a cut of H_{3-i} (as H_1 and H_2 are equivalent). It follows that every cut of G_i is a cut of G_{3-i}. Therefore, by Remark 21, every cut of (G_i,T_i) is even. Therefore $T_i = \emptyset$. It follows by Theorem 19 that $\Sigma_{3-i} = \emptyset$, in particular M is graphic. Hence, N is graphic as well, contradicting our hypothesis.

It remains to show that $\mathbb{F}'$ can be partitioned into exactly two equivalence classes. Suppose, for a contradiction, this is not the case. Then, there exist, for $i = 1,2,3$, $(G_i,\Sigma_i) \in \mathbb{F}'$, where G_1,G_2 and G_3 are pairwise inequivalent. For $i = 1,2,3$ let T_i denote the endpoint(s) of Ω in G_i. It follows from the argument in the previous paragraph that, $|T_1| = |T_2| = 2$ and that $\operatorname{ecut}(G_1,T_1) = \operatorname{ecut}(G_2,T_2)$. Similarly, we have that $|T_2| = |T_3| = 2$ and that $\operatorname{ecut}(G_2,T_2) = \operatorname{ecut}(G_3,T_3)$. For $i = 1,2$, let $v_i \in T_i$

and let $D_i := \delta_{G_i}(v_i)$. Theorem 19 applied to the pair G_1, G_3 implies that D_1 is a signature of (G_3, Σ_3). Similarly, D_2 is a signature of (G_3, Σ_3). Hence, $D_1 \triangle D_2$ is a cut of G_3. As $\Omega \notin D_1 \triangle D_2$ we have that $D_1 \triangle D_2$ is an even cut of (G_3, T_3). It follows that $D_1 \triangle D_2$ is an even cut of (G_1, T_1). Hence, $D_1 \triangle (D_1 \triangle D_2) = D_2$ is an odd cut of G_1. But now Remark 21 implies that G_1 and G_2 are equivalent, a contradiction. □

We are now ready for the main result of this section,

Proof of Lemma 15. Recall that in the definition of "row stable" representations, we assumed that M has no loops, no co-loops and that N is not graphic. Hence, if the result does not hold we must have N, M, $\mathbb{F}$, $\mathbb{F}' = \mathbb{F}_1 \cup \mathbb{F}_2$ and Ω as in Lemma 24 and Ω-siblings $(G_1, \Sigma_1) \in \mathbb{F}_1$ and $(G_2, \Sigma_2) \in \mathbb{F}_2$ that arise from some template $\mathbb{T} = (H_1, v_1,\ \alpha_1,\ H_2, v_2, \alpha_2)$. As the siblings are Ω-siblings, we have $H_i = G_i/\Omega$ for $i = 1, 2$. Because of Remark 20 we may assume (after possibly a signature exchange) that $\Sigma_1 = \Sigma_2 = \alpha_1 \triangle \alpha_2$. Hence, $(H_1, \alpha_1 \triangle \alpha_2), (H_2, \alpha_1 \triangle \alpha_2) \in \mathbb{F}$. It follows that (H_1, v_1) and (H_2, v_2) is an intercepting pair of $(H_1, \alpha_1 \triangle \alpha_2)$, contradicting our hypothesis. □

4 A characterization of split siblings

We will rely on Lemma 24 to prove the missing Lemma 16. The key to the proof is a theorem that characterizes split-siblings. Before we can state the theorem we need to understand 3-connected even cycle matroids.

4.1 Connectivity

Let G be a graph and let $X \subseteq E(G)$. The set X is a *k-separation* of G if $\min\{|X|, |\bar{X}|\} \geq k$, $|\mathscr{B}_G(X)| = k$ and both $G[X]$ and $G[\bar{X}]$ are connected. A graph G is k-connected if it has no r-separations for any $r < k$. Recall that, a signed graph (G, Σ) is *bipartite* if all cycles are even.

Proposition 25. *Suppose that* $\text{ecycle}(G, \Sigma)$ *is* 3*-connected. Then*

(1) $|\text{loop}(G)| \leq 1$ *and if* $e \in \text{loop}(G)$ *then* $e \in \Sigma$*;*

(2) $G \setminus \text{loop}(G)$ *is* 2*-connected;*

(3) if G has a 2*-separation X, then* $(G[X], \Sigma \cap X)$ *and* $(G[\bar{X}], \Sigma \cap \bar{X})$ *are both non-bipartite.*

To prove the previous theorem we require a definition and a preliminary result. Let (G, Σ) be a signed graph and $X \subseteq E(G)$. Then X is a *k-(i, j)-separation* of (G, Σ), where $i, j \in \{0, 1\}$, if the following hold:

(a) X is a k-separation of G;

(b) $i=0$ when $(G[X],\Sigma\cap X)$ is bipartite and $i=1$ otherwise;

(c) $j=0$ when $(G[\bar{X}],\Sigma\cap\bar{X})$ is bipartite and $j=1$ otherwise.

Lemma 26. *Let (G,Σ) be a non-bipartite signed graph and $M_S := \mathrm{ecycle}(G,\Sigma)$. For every k-(i,j)-separation X of (G,Σ), we have $\lambda_{M_S}(X) = k+i+j-1$.*

Proof. Let r be the rank function of $M := \mathrm{cycle}(G)$ and r_S be the rank function of M_S. As (G,Σ) is non-bipartite, a basis for M_S consists of a spanning tree B of G plus an edge $e\in\bar{B}$ that forms a Σ-odd circuit with elements in B. Hence $r_S(M_S)=r(M)+1$. Similarly, if $(G[X],\Sigma\cap X)$ (respectively $(G[\bar{X}],\Sigma\cap\bar{X})$) is non-bipartite, then the rank of X (respectively $\bar{X}$) in M_S is one more that in M, otherwise the rank of X (respectively $\bar{X}$) is the same in both matroids. Thus $r_S(X)=r(X)+i$ and $r_S(\bar{X})=r(\bar{X})+j$. Hence

$$\begin{aligned}\lambda_{M_S}(X) &= r_S(X)+r_S(\bar{X})-r_S(M_S)+1\\ &= [r(X)+i]+[r(\bar{X})+j]-[r(M)+1]+1\\ &= \lambda_M(X)+i+j-1\\ &= k+i+j-1\end{aligned}$$ $\square$

Proof of Proposition 25. Let $M:=\mathrm{ecycle}(G,\Sigma)$. As M is 3-connected, it has no loops, no co-loops and no parallel elements. We may assume that (G,Σ) is non-bipartite, for otherwise $M=\mathrm{cycle}(G)$ and G is 3-connected. **(1)** Let e be a loop of G. Then $e\in\Sigma$ for otherwise e would be a loop of M. There do not exist distinct loops e,f of G, for otherwise $\{e,f\}$ would be a circuit of M and e,f would be in parallel in M. **(2)** Suppose that X is a 1-(i,j)-separation of (G,Σ). By Lemma 26, $\lambda_M(X)=1+i+j-1\le 2$. As M is 3-connected, X is not a 2-separation; hence either $|X|=1$ or $|\bar{X}|=1$. The single element in X (or $\bar{X}$) is not a bridge of G, for otherwise it is a co-loop of M. Hence X or $\bar{X}$ is a loop of G. **(3)** Suppose that X is a 2-(i,j)-separation of (G,Σ). As M is 3-connected, $\lambda_M(X)\ge 3$. By Lemma 26, $2+i+j-1\ge 3$, hence $i=j=1$. $\square$

We say that $\mathbb{S}=(X_1,\dots,X_k)$ is a *w-sequence* of G if, for all $i\in[k]$, X_i is a 2-separation of the graph obtained from G by performing Whitney-flips on $X_1,\dots,X_{i-1}$ (in this order). We denote by $\mathrm{W}_{\mathrm{flip}}[G,\mathbb{S}]$ the graph obtained from G by performing Whitney-flips on $X_1,\dots,X_k$ (in this order). For our purpose the position of loops is irrelevant. Hence we will assume that loops form distinct components of the graph. Therefore, if G and G' are equivalent graphs that are 2-connected, except for possible loops, then $G'=\mathrm{W}_{\mathrm{flip}}[G,\mathbb{S}]$ for some w-sequence $\mathbb{S}$ of G.

Consider a split-template $(H_1,v_1,\alpha_1,H_2,v_2,\alpha_2)$. If H_1 and H_2 are 2-connected, except for possible loops, we have that $H_2=\mathrm{W}_{\mathrm{flip}}[H_1,\mathbb{S}]$ for some w-sequence $\mathbb{S}$. (This occurs, by Proposition 25, when for instance $\mathrm{ecycle}(H_1,\alpha_1\triangle\alpha_2)$ is 3-connected.) In this case we slightly abuse terminology and say that $(H_1,v_1,\,\alpha_1,\,H_2,\,v_2,\,\alpha_2,\mathbb{S})$ is a split-template.

4.2 The statement of the theorem

The next result gives a structural characterization of Ω-split siblings.

Theorem 27. *Let* (G_1,Σ_1) *and* (G_2,Σ_2) *be* Ω*-split siblings. Suppose* $\mathrm{ecycle}(G_1,\Sigma_1)$ *and* $\mathrm{ecycle}(G_1,\Sigma_1)/\Omega$ *are both* 3*-connected. Then* (G_1,Σ_1) *and* (G_2,Σ_2) *are either simple siblings or nova siblings.*

Note, in the statement of the theorem we have by definition that $\mathrm{ecycle}(G_1,\Sigma_1) = \mathrm{ecycle}(G_2,\Sigma_2)$. Hence, in particular $\mathrm{ecycle}(G_2,\Sigma_2)$ and $\mathrm{ecycle}(G_2,\Sigma_2)/\Omega$ are both 3-connected as well. We need to define the terms "simple siblings" and "nova siblings". We begin by defining a more restrictive notion, namely simple and nova *twins*.

4.2.1 Simple twins

Consider a split-template $\mathbb{T} = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, \mathbb{S})$. If $\mathbb{S} = \emptyset$, i.e. $H_1 = H_2$, then $\mathbb{T}$ is *simple* and (G_1,Σ_1) and (G_2,Σ_2) arising from $\mathbb{T}$ are *simple twins*. By Remark 20, we may assume that $\Sigma_1 = \Sigma_2 = \alpha_1 \triangle \alpha_2$. Suppose that vertex v_1 of H_1 gets split into vertices v_1^- and v_1^+ of G_1 according to α_1. Then $\alpha_1 \subseteq \delta_{G_1}(v_1^-)$ and $\alpha_2 \subseteq \delta_{G_1}(v_2)$. Hence, v_1^- and v_2 form a blocking pair of (G_1,Σ_1). Thus,

Remark 28. *Simple twins have blocking pairs.*

4.2.2 Nova twins

Let (H,Σ) be a signed graph with distinct vertices s_1 and s_2. Suppose for $i = 1,2$, there exists an odd circuit C_i using s_i and avoiding s_{3-i}. If either, C_1 and C_2 intersect in a path, or $V(C_1) \cap V(C_2) = \emptyset$ and there exists a path P with ends $u_i \in V(C_i) - \{s_i\}$, for $i = 1,2$, such that $V(P) \cap \big(V(C_1) \cup V(C_2)\big) = \{u_1,u_2\}$, then we say that there exists an $\{s_1,s_2\}$-handcuff in (H,Σ). Let (H,Σ) be a signed graph and consider a 2-separation X of H where $\mathscr{B}(X) = \{s_1,s_2\}$. We say that X is a *handcuff-separation* if $\{s_1,s_2\}$ is a blocking pair of $(H[X],\Sigma\cap X)$ and there exists an $\{s_1,s_2\}$-handcuff in $(H[X],\Sigma\cap X)$.

A family $\mathbb{S} = \{X_1,\dots,X_k\}$ of sets of edges of a graph H is a *w-star* with *center* v if

(a) $X_i \cap X_j = \emptyset$, for all distinct $i,j \in [k]$;

(b) there exist distinct $v, w_1, \dots, w_k \in V(H)$ such that $\mathscr{B}(X_i) = \{v, w_i\}$, for all $i \in [k]$;

(c) no edge with ends v, w_i is in X_i, for all $i \in [k]$.

Consider a w-sequence $\mathbb{S} = (X_1,\dots,X_k)$ of a graph G where $X_i \cap X_j = \emptyset$ for all distinct $i,j \in [k]$. If $i_1,\dots,i_k$ is a permutation of $[k]$, then $\mathbb{S}' := (X_{i_1},\dots,X_{i_k})$ is a w-sequence as well and $\mathrm{W}_{\text{flip}}[G,\mathbb{S}] = \mathrm{W}_{\text{flip}}[G,\mathbb{S}']$. Thus we can think of a w-star as a special type of w-sequence (where we can order the sets in an arbitrary way).

A split-template $\mathbb{T} = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, \mathbb{S})$ is *nova* if, for $i = 1,2$:

(a) $\mathbb{S}$ is a w-star of H_i with center v_i, and

(b) all $X' \subseteq X \in \mathbb{S}$ with $\mathscr{B}_{H_i}(X') = \mathscr{B}_{H_i}(X)$ are handcuff-separations of $(H_i, \alpha_1 \triangle \alpha_2)$.

We say that (G_1,Σ_1) and (G_2,Σ_2) arising from $\mathbb{T}$ are *nova twins*. By Remark 20 we may assume that $\Sigma_1 = \Sigma_2 = \alpha_1 \triangle \alpha_2$. This construction is illustrated in Figure 2 for the case where $k = |\mathbb{S}| = 2$. The signed graph on the left (resp. right) represents (G_1,Σ_1) (resp. (G_2,Σ_2)). The arrows indicate how each piece is flipped to obtained (G_2,Σ_2) from (G_1,Σ_1). Shaded regions around a vertex v indicate the odd edges incident to v. For $i = 1,2$, recall that $\alpha_i \subseteq \delta_{H_i}(v_i)$, we denote by $\bar{\alpha}_i$ the set $\delta_{H_i}(v_i) - \alpha_i$.

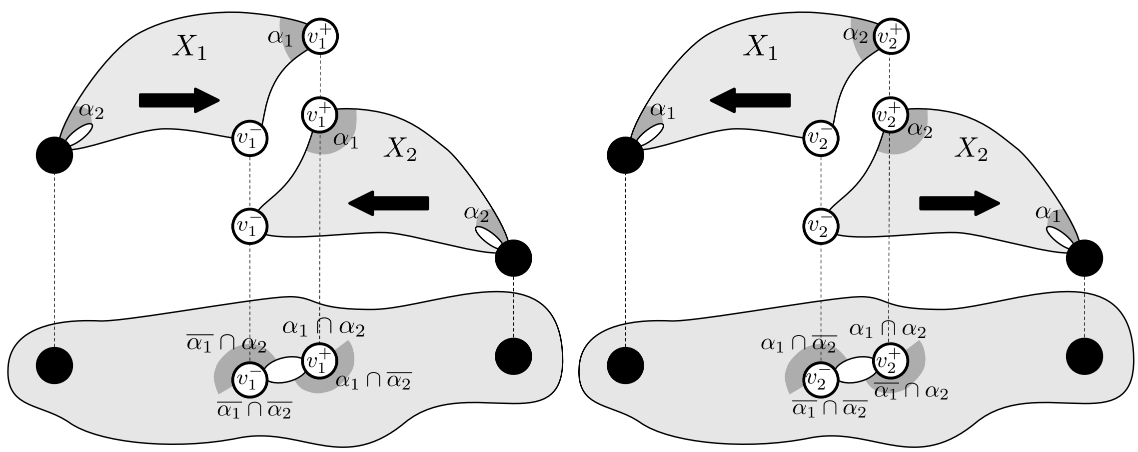

Figure 2: Nova twins

.

4.2.3 From twins to siblings

We say that (G_1,Σ_1) and (G_2,Σ_2) are simple (respectively nova) *siblings* if, for $i = 1,2$, there exists (G'_i,Σ'_i) equivalent to (G_i,Σ_i) such that (G'_1,Σ'_1) and (G'_2,Σ'_2) are simple (respectively nova) twins.

4.2.4 A corollary

Using the nova construction we can find distinct representations of an even cycle matroid with an arbitrary number of vertex disjoint odd circuits.

Proof of Remark 3. Let $\mathbb{T} = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, \mathbb{S})$ be a split-template which is nova. Let (G_1,Σ_1) and (G_2,Σ_2) be the siblings arising from $\mathbb{T}$. Because of Remark 20, we may assume that $\Sigma_1 = \Sigma_2 = \alpha_1 \triangle \alpha_2$. Suppose that $\mathbb{S} = (X_1,\ldots,X_k)$ for some integer k. Because of (b) (in the definition of nova), for every $j \in [k]$, there exists an odd circuit $C_j \subseteq X_j$ of (H_1,Σ_1) avoiding v_1. In particular, C_j remains an odd circuit of (G_1,Σ_1). Thus odd circuits $C_1,\ldots,C_k$ of (G_1,Σ_1) are pairwise vertex disjoint. Moreover, it is easy to select H_1 so that the only 2-separations of H_1 are given by $\mathbb{S}$. Then G_1 is 3-connected.

Hence, (1) holds with $(G,\Sigma)=(G_1,\Sigma_1)$. Moreover, $\text{ecycle}(G_1,\Sigma_1)=\text{ecycle}(G_2,\Sigma_2)$, thus (2) holds as required. $\square$

4.3 An outline of the proof of Theorem 27

We say that split-templates:

$$\mathbb{T}=(H_1,v_1,\alpha_1,H_2,v_2,\alpha_2,\mathbb{S}) \quad \text{and} \quad \mathbb{T}'=(H_1',v_1',\alpha_1',H_2',v_2',\alpha_2',\mathbb{S}') \tag{1}$$

are *compatible* if

(a) H_i and H_i' are equivalent, for $i=1,2$, and

(b) $\alpha_i \triangle \alpha_i'$ forms a cut of H_1 (hence of H_2) for $i=1,2$.

Lemma 29. *Let $\mathbb{T}$ and $\mathbb{T}'$ be compatible split-templates. Let (G_1,Σ_1) and (G_2,Σ_2) be siblings arising from $\mathbb{T}$ and let (G_1',Σ_1'), (G_2',Σ_2') be arising from $\mathbb{T}'$. Then, for $i=1,2$, (G_i,Σ_i) and (G_i',Σ_i') are equivalent.*

Proof. Let us assume that $\mathbb{T}$ and $\mathbb{T}'$ are as described in (1). Then G_1 (resp. G_1') is obtained from H_1 (resp. H_1') by splitting v_1 (resp. v_1') according to α_1 (resp. α_1'). It follows that every cut of G_1 is either a cut of H_1 or is equal to α_1. Similarly, every cut of G_1' is either a cut of H_1' or is equal to α_1'. Hence, (viewing cuts as vector spaces),

$$\text{cut}(G_1)=\text{span}\big(\text{cut}(H_1)\cup\{\alpha_1\}\big) \quad \text{and} \quad \text{cut}(G_1')=\text{span}\big(\text{cut}(H_1')\cup\{\alpha_1'\}\big). \tag{$\star$}$$

By condition (a) of equivalent templates, H_1 and H_1' are equivalent. Hence, $\text{cycle}(H_1)=\text{cycle}(H_1')$, and therefore $\text{cut}(H_1)=\text{cut}(H_1')$. By condition (b) of equivalent templates, $\alpha_1\triangle\alpha_1'\in\text{cut}(H_1)$. These two statements and $(\star)$ imply that $\text{cut}(G_1)=\text{cut}(G_1')$, and therefore that $\text{cycle}(G_1)=\text{cycle}(G_1')$. It follows from Theorem 1 that G_1 and G_1' are equivalent. Remark 20 implies that $\text{ecycle}(G_1,\alpha_1\triangle\alpha_2)=\text{ecycle}(G_2,\alpha_1\triangle\alpha_2)$. By Theorem 19, $\Sigma_1=\alpha_1\triangle\alpha_2\triangle B$ where B is a cut of G_1. Similarly, $\Sigma_1'=\alpha_1'\triangle\alpha_2'\triangle B'$ where B' is a cut of G_1', hence of G_1. It follows from condition (b) that $\Sigma_1\triangle\Sigma_1'$ is a cut of G_1 and G_1', hence that (G_1,Σ_1) and (G_1',Σ_1') are equivalent. Similarly, we can prove that (G_2,Σ_2) and (G_2',Σ_2') are equivalent. $\square$

We will postpone the proof of the following Lemma until Section 4.4

Lemma 30. *Every split-template has a compatible split-template which is simple or nova.*

Proof of Theorem 27. By definition, (G_1,Σ_1) and (G_2,Σ_2) arise from a split-template $(H_1,v_1,\alpha_1,H_2,v_2,\alpha_2)$. Since $\text{ecycle}(G_1,\Sigma_1)/\Omega$ is 3-connected and since $G_1/\Omega=H_1$, it follows by Proposition 25 that H_1 is 2-connected, up to loops. Similarly, H_2 is 2-connected, up to loops. In particular, $H_2=\text{W}_{\text{flip}}[H_1,\mathbb{S}]$ for some w-sequence $\mathbb{S}$. Denote by $\mathbb{T}$ the template $(H_1,v_1,\alpha_1,H_2,v_2,\alpha_2,\mathbb{S})$. Lemma 30 implies that there exists a split-template $\mathbb{T}'$ which is simple or nova and compatible with $\mathbb{T}$. Let (G_1', Σ_1') and (G_2', Σ_2') be the siblings arising from $\mathbb{T}'$. By definition (G_1', Σ_1') and (G_2', Σ_2') are simple twins or nova twins. By Lemma 29, for $i = 1, 2$, (G_i', Σ_i') is equivalent to (G_i, Σ_i). Hence, (G_1, Σ_1) and (G_2, Σ_2) are simple or nova siblings. □

4.4 Proof of Lemma 30

The following result on 2-separations is postponed until Section 4.5.

Proposition 31. *Consider* 2*-connected equivalent graphs G and G' and let $z \in V(G)$ and $z' \in V(G')$. There exist w-sequences $\mathbb{L}$ of G, $\mathbb{L}'$ of G' and graphs H and H' such that:*

(1) $H = \mathrm{W}_{\mathit{flip}}[G, \mathbb{L}]$, where $z \notin \mathscr{B}_G(X)$, for all $X \in \mathbb{L}$;

(2) $H' = \mathrm{W}_{\mathit{flip}}[G', \mathbb{L}']$, where $z' \notin \mathscr{B}_{G'}(X)$, for all $X \in \mathbb{L}'$; and

(3) $H' = \mathrm{W}_{\mathit{flip}}[H, \mathbb{S}]$,

where $\mathbb{S}$ is a w-star of H with center z (or equivalently a w-star of H' with center z').

Given a graph H and $U \subseteq V(H)$ we denote by $H - U$ the graph obtained from H by deleting all vertices in U. We write $H - u$ as shorthand for $H - \{u\}$. Let G be a graph with disjoint vertex sets A and B. An *$A - B$ path* is a path of H with one endpoint in A and one endpoint in B. We use "$a - b$ path" as shorthand for "$\{a\} - \{b\}$ path" and similarly, "$a - B$ path" as shorthand for "$\{a\} - B$ path". Given a graph G and $X \subseteq E(G)$, we denote by $\mathscr{I}_G(X)$ the set $V_G(X) - \mathscr{B}_B(X)$.

Next we give sufficient conditions for a 2-separation to be a handcuff-separation.

Remark 32. *Let (G, Σ) be a signed graph where* $\mathrm{ecycle}(G, \Sigma)$ *is* 3*-connected. Then a* 2*-separation X of G with $\mathscr{B}(X) = \{s_1, s_2\}$ is a handcuff-separation if the following conditions hold.*

(a) $\Sigma \cap X \subseteq \delta(s_1) \cup \delta(s_2)$;

(b) For $i = 1, 2$ the sets $X \cap \delta(s_i) \cap \Sigma$ and $\big(X \cap \delta(s_i)\big) - \Sigma$ are non-empty;

(c) There does not exists $X' \subset X$ where $\mathscr{B}(X') = \mathscr{B}(X)$.

Proof.

Claim. *For all distinct vertices u, v in $\mathscr{I}(X)$ there exists a $u - v$ path in $G[X] - \{s_1, s_2\}$.*

Proof. Otherwise there exists components $G[Y_1]$, $G[Y_2]$ of $G[X] - \{s_1, s_2\}$ where $u \in V(Y_1)$ and $v \in V(Y_2)$. Let $X' := Y_1 \cup \{(s_1, u) \in E(G) : u \in V(Y_1)\} \cup \{(s_2, u) \in E(G) : u \in V(Y_1)\}$. Then $\mathscr{B}(X') \subseteq \{s_1, s_2\}$. Moreover, equality holds since $\mathrm{ecycle}(G, \Sigma)$ is 3-connected (Proposition 25). Hence, $\mathscr{B}(X') = \mathscr{B}(X)$, contradicting hypothesis (c). ◇

By hypothesis (b) there exist edges $e=(s_1,u)\in X\cap\Sigma$ and $f=(s_1,v)\in X-\Sigma$. We may assume that $u\neq s_2$, for otherwise $X':=\{e\}$ satisfies $\mathscr{B}(X')=\mathscr{B}(X)$, contradicting hypothesis (c). Similarly, $v\neq s_2$. The Claim implies that there exists a $u-v$ path P in $G[X]-\{s_1,s_2\}$. Then $C_1:=\{e,f\}\cup P$ is an odd circuit of $(G[X],\Sigma\cap X)$ avoiding s_2. Similarly, there exists an odd circuit C_2 of $(G[X],\Sigma\cap X)$ avoiding s_1. Suppose $V(C_1)\cap V(C_2)\neq\emptyset$. Because of hypothesis (a), we may assume (after possibly redefining C_1,C_2) that C_1 and C_2 intersect in a path, in which case X is a handcuff-separation as required. Suppose $V(C_1)\cap V(C_2)=\emptyset$. Then for $i=1,2$ let u_i denote an arbitrary vertex of C_i that is disjoint from s_i. By the Claim there exists a u_1-u_2 path P of $G[X]-\{s_1,s_2\}$. If among all such path we pick one with as few edges as possible, then $V(P)\cap\big(V(C_1)\cup V(C_2)\big)=\{u_1,u_2\}$, in which case X is a handcuff-separation as required. $\square$

Proof of Lemma 30. Let $\mathbb{T}:=(H_1,v_1,\alpha_1,H_2,v_2,\alpha_2,\mathbb{S})$ be a split-template. Proposition 31 implies that there exist w-sequences $\mathbb{L}_1$ of H_1, $\mathbb{L}_2$ of H_2, such that

- $H_1'=\mathrm{W}_{\text{flip}}[H_1,\mathbb{L}_1]$, where $v_1\notin\mathscr{B}_{H_1}(X)$ for all $X\in\mathbb{L}_1$;
- $H_2'=\mathrm{W}_{\text{flip}}[H_2,\mathbb{L}_2]$, where $v_2\notin\mathscr{B}_{H_2}(X)$ for all $X\in\mathbb{L}_2$;
- $H_2'=\mathrm{W}_{\text{flip}}[H_1',\mathbb{S}]$

where $\mathbb{S}$ is a w-star of H_1' with center v_1. Observe that $(H_1',v_1,\alpha_1,H_2',v_2,\alpha_2,\mathbb{S})$, is a template that is compatible with $\mathbb{T}$. Thus we can choose, among all templates $\mathbb{T}'=(H_1',v_1',\alpha_1',H_2',v_2',\alpha_2',\mathbb{S}')$ where $\mathbb{T}'$ is compatible with $\mathbb{T}$ and where $\mathbb{S}'$ is a w-star of H_1' with center v_1', one that minimizes

$$\sum_{X\in\mathbb{S}'}|X|. \tag{$\star$}$$

We may assume $\mathbb{S}'\neq\emptyset$ for otherwise $\mathbb{T}'$ is a simple template. We will show that $\mathbb{T}'$ is a nova template.

We need to show that conditions (a) and (b) of nova templates hold (see Section 4.2.2). It suffices to consider the case $i=1$ as the proof for the case $i=2$ is similar. Since $\mathbb{S}'$ is a w-star of H_1', condition (a) holds. Let $X\in\mathbb{S}'$ and let X' be an inclusion-wise minimal set $X'\subseteq X$ such that $\mathscr{B}_{H_1'}(X)=\mathscr{B}_{H_1'}(X')$. Denote by p the vertex in $\mathscr{B}_{H_1'}(X)$ that is distinct from v_1'. Since $\mathbb{S}'$ is a w-star of H_1' with center v_1' and a w-star of H_2' with center v_2', edges in $\alpha_1'\cap X'$ are incident to v_1' in H_1' and edges in $\alpha_2'\cap X'$ are incident to v_2' in H_2', i.e. to vertex p in H_1'. It follows, in particular, that v_1',p is a blocking pair of $\big(H_1'[X'],(\alpha_1'\triangle\alpha_2')\cap X'\big)$. Hence, to show that condition (b) holds, we need to verify that there exists a $\{v_1',p\}$-handcuff in $\big(H_1'[X'],(\alpha_1'\triangle\alpha_2')\cap X'\big)$.

Define the following sets,

$$A_1:=X'\cap\delta_{H_1'}(v_1')\cap\alpha_1', \qquad A_2:=\big(X'\cap\delta_{H_1'}(v_1')\big)-\alpha_1'$$
$$A_3:=X'\cap\delta_{H_1'}(p)\cap\alpha_2', \qquad A_4:=\big(X'\cap\delta_{H_1'}(p)\big)-\alpha_2'.$$

Since $\mathbb{S}'$ is a w-star of H_1' there is no edge $e = (v_1', p) \in X'$. In particular, $(\alpha_1' \cup \alpha_2') \cap X'$ is a signature of $\big(H_1'[X'], (\alpha_1' \triangle \alpha_2') \cap X'\big)$. Therefore, because of Remark 32, it suffices to prove that each of A_1, A_2, A_3, A_4 are non-empty. Observe that, A_3, A_4 play the same role in H_2' as the sets A_1, A_2 in H_1'. Hence, it will suffice to show that $A_1, A_2 \neq \emptyset$.

Suppose for a contradiction that $A_1 = \emptyset$. Then let $\hat{H}$ be obtained from H_1' by a Whitney-flip on X' and let $\hat{v}$ denote the vertex of $\hat{H}$ (that used to correspond to vertex v_1 in H_1') that is incident to all edges in α_1'. Let $\hat{\mathbb{S}}$ be obtained from $\mathbb{S}'$ by replacing the set X with $X - X'$ if $X' \subset X$ or by removing the set X if $X = X'$. Then $(\hat{H}, \hat{v}, \alpha_1', H_2', v_2', \alpha_2', \hat{\mathbb{S}})$ is a template that is compatible with $\mathbb{T}$ but that contradicts our choice ($\star$). Suppose for a contradiction that $A_2 = \emptyset$. Define $\hat{\alpha} := \alpha_1' \triangle \delta_{H_1'}(v_1)$. Then $(H_1', v_1', \hat{\alpha}, H_2', v_2', \alpha_2', \mathbb{S}')$ is a template that is compatible with $\mathbb{T}$. Since no edge of X' incident to v_1' is in $\hat{\alpha}$ it corresponds to the case $A_1 = \emptyset$ and we obtain a contradiction as previously. $\square$

4.5 Proof of Proposition 31

A pair of sets X and Y are *crossing* if all of $X \cap Y, X - Y, Y - X$ and $\bar{X} \cap \bar{Y}$ are non-empty. A sequence $\mathbb{S}$ is *non-crossing* if no two sets of $\mathbb{S}$ are crossing. Note that if $\mathbb{S}$ is a non-crossing w-sequence $(X_1, \ldots, X_k)$ of a graph G, then for any permutation $i_1, \ldots, i_k$ of $[k]$, $\mathbb{S}' := (X_{i_1}, \ldots, X_{i_k})$ is a w-sequence of G and $\mathrm{W}_{\text{flip}}[G, \mathbb{S}] = \mathrm{W}_{\text{flip}}[G, \mathbb{S}']$. Hence, we can view a non-crossing w-sequence sequence as a family of sets.

We say that a family $\mathbb{F}$ of sets satisfies the *inclusion property* if there does not exists $X_1, X_2, X_3 \in \mathbb{F}$ such that $\widetilde{X}_1 \subset \widetilde{X}_2 \subset \widetilde{X}_3$ where for $i \in [3]$, $\widetilde{X}_i$ denotes either X_i or $\bar{X}_i$.

Remark 33. *Suppose that $\mathbb{F}$ is a non-crossing family of sets with the inclusion property. Then, after possibly replacing some of the sets of $\mathbb{F}$ by their complements, the sets of $\mathbb{F}$ are pairwise disjoint.*

Proof. Let E denote the ground set of the sets in $\mathbb{F}$. Let $k := |\mathbb{F}|$ and let us proceed by induction on k. The result is trivial if $k = 1$. Suppose $k = 2$, i.e. $\mathbb{F} = \{X_1, X_2\}$. Then either: (a1) $X_1 \cap X_2 = \emptyset$, or (a2) $X_i \subseteq X_{3-i}$, for $i \in [2]$, or (a3) $X_1 \cap X_2, X_1 - X_2, X_2 - X_1$ are all non-empty. For (a2) replace X_{3-i} by $\bar{X}_{3-i}$. For (a3) as $\mathbb{F}$ is non-crossing $X_1 \cup X_2 = E$ and replace X_1 by $\bar{X}_1$ and X_2 by $\bar{X}_2$.

Assume the results holds for some $k \geq 2$. Suppose $\mathbb{F} = \{X_1, \ldots, X_{k+1}\}$. We may assume by induction that $X_1, \ldots, X_k$ are pairwise disjoint. Then for any $i \in [k]$ either (b1) $X_{k+1} \cap X_i$ is equal to X_i or the empty set, or (b2) $X_{k+1} \subset X_i$, or (b3) $X_{k+1} \cap X_i, X_{k+1} - X_i, X_i - X_{k+1}$ are all non-empty. For (b2), let $j \in [k]$, $j \neq i$, then $X_{k+1} \subset X_i \subset \bar{X}_j$, contradicting the inclusion property. For (b3) as $\mathbb{F}$ is non-crossing, $X_{k+1} \cup X_i = E$. Then after replacing X_{k+1} by $\bar{X}_{k+1}$ we are in case (b2). Thus either, (c1) $X_{k+1} \cap X_i = \emptyset$ for all $i \in [k]$, or (c2) $X_{k+1} \supset X_i$ for all $i \in [k]$, or (c3) there exists $i, j \in [k]$ such that $X_i \subset X_{k+1}$ and $X_j \cap X_{k+1} = \emptyset$. For (c2), replace X_{k+1} by $\bar{X}_{k+1}$. For (c3), $X_i \subset X_{k+1} \subset \bar{X}_j$, contradicting the inclusion property. $\square$

Lemma 34. *Let H and H' be 2-connected equivalent graphs with $H' = \mathrm{W}_{\text{flip}}[H, \mathbb{S}]$ for some non-crossing w-sequence $\mathbb{S}$. Suppose that there exist vertices z in $V(H)$ and z' in $V(H')$ such that $z \in \mathscr{B}_H(X)$*

and $z' \in \mathscr{B}_{H'}(X)$ for every $X \in \mathbb{S}$. Then $H' = \mathrm{W}_{\mathit{flip}}[H,\mathbb{S}']$ for some $\mathbb{S}'$ which is a w-star of H (resp. H') with center z (resp. z').

Proof. We may assume that there does not exists $X_1, X_2 \in \mathbb{S}$ with $\mathscr{B}_H(X_1) = \mathscr{B}_H(X_2)$, for otherwise we can replace in $\mathbb{S}$ the sets X_1 and X_2 by $X_1 \triangle X_2$. If any $X \in \mathbb{S}$ contains an edge e where the ends of e are $\mathscr{B}_H(X)$, we may replace X by $X - \{e\}$. Thus properties (b) and (c) of w-star holds (see Section 4.2.2). Property (a) will follow from the following Claim and Remark 33.

Claim. *$\mathbb{S}$ satisfies the inclusion property.*

Proof. Otherwise, we may assume, that there exists $X_1, X_2, X_3 \in \mathbb{S}$ such that $X_1 \subset X_2 \subset X_3$. Moreover, after possibly redefining the sets X_1, X_2, X_3, that

(i) X_2 is the unique set $X \in \mathbb{S}$ such that $X_1 \subset X \subset X_3$.

For $i = 1,2,3$, denote by v_i the vertex of H for which $\mathscr{B}_H(X_i) = \{z, v_i\}$. Since H is 2-connected. There exists a non-empty $v_1 - v_2$ path P_1 in $X_2 - X_1$ avoiding z and there exists a non-empty $v_2 - v_3$ path P_2 in $X_3 - X_2$ avoiding z. Since $\mathbb{S}$ is non-crossing, we may assume that X_1, X_2, X_3 are the first three elements in the sequence $\mathbb{S}$. Let $\hat{H} = \mathrm{W}_{\mathrm{flip}}[H, (X_1, X_2, X_3)]$. Observe that for the graph $\hat{H}$,

(ii) P_1 and P_2 are vertex disjoint $\mathscr{B}(X_1) - \mathscr{B}(X_3)$ paths of $\hat{H}$.

Since, X_4 is non-crossing with X_1, X_2 and X_3, vertices of $\mathscr{B}_{\hat{H}}(X_4)$ are in one of the following sets, (a) $V_{\hat{H}}(X_1)$, (b) $V_{\hat{H}}(\bar{X}_3)$, (c) $V_{\hat{H}}(X_2 - X_1)$ or $V_{\hat{H}}(X_3 - X_2)$. Moreover, for (c) $\mathscr{B}_{\hat{H}}(X_4)$ does not intersect both $V_{\hat{H}}(P_1)$ and $V_{\hat{H}}(P_2)$, for otherwise we contradict (i). Hence, for all all cases (ii) holds for $\mathrm{W}_{\mathrm{flip}}[\hat{H}, (X_4)] = \mathrm{W}_{\mathrm{flip}}[H, (X_1, X_2, X_3, X_4)]$. Repeating the argument for every $X \in \mathbb{S}$ where $X \neq X_1, X_2, X_3, X_4$ we deduce that (ii) holds for H' as well. In particular, it implies that $\mathscr{B}_{H'}(X_1) \cap \mathscr{B}_{H'}(X_3) = \emptyset$, a contradiction as $z' \in \mathscr{B}_{H'}(X_1) \cap \mathscr{B}_{H'}(X_3)$. □

In the interest of brevity we omit the proof of the following “folklore” result.

Lemma 35. *Let G and G' be 2-connected equivalent graphs and let $z \in V(G)$. Then there exist a w-sequence $\mathbb{S}_1$ of G and a graph H with a non-crossing w-sequence $\mathbb{S}_2$ such that:*

(1) $H = \mathrm{W}_{\mathit{flip}}[G, \mathbb{S}_1]$, where $z \notin \mathscr{B}_G(X)$, for all $X \in \mathbb{S}_1$; and

(2) $G' = \mathrm{W}_{\mathit{flip}}[H, \mathbb{S}_2]$, where $z \in \mathscr{B}_G(X)$, for all $X \in \mathbb{S}_2$.

See [5](lemma 5.8) for a proof.

Proof of Proposition 31. Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be the w-sequences given by Lemma 35. Denote by $\mathbb{L}$ the sequence $\mathbb{S}_1$ and let $H = \mathrm{W}_{\mathrm{flip}}[G, \mathbb{L}]$. Partition $\mathbb{S}_2$ into sequences $\mathbb{S}$ and $\mathbb{L}'$ (with sets in an arbitrary order) where $X \in \mathbb{S}$ if and only if $z' \in \mathscr{B}_{G'}(X)$. Let $H' = \mathrm{W}_{\mathrm{flip}}[G', \mathbb{L}']$. Since $\mathbb{S}_2$ is non-crossing, $H' = \mathrm{W}_{\mathrm{flip}}[H, \mathbb{S}]$. Finally, Lemma 34 implies that $\mathbb{S}$ is a w-star of H. □

5 Row extensions and blocking pairs

5.1 Outline of the proof

The goal of this section is to prove Lemma 16. We will consider a number of results with a common set of hypotheses that we state next.

Hypothesis 36. *$\mathbb{T} = (H, v, \alpha, H', v', \alpha', \mathbb{S})$ is a split template that is nova. G and G' are graphs that both contain an edge Ω where $H = G/\Omega$, $H' = G'/\Omega$. In addition, $(G, \alpha \triangle \alpha')$ and $(G', \alpha \triangle \alpha')$ are nova twins arising from $\mathbb{T}$. Moreover,*

(h1) *no signed graph equivalent to $(H, \alpha \triangle \alpha')$, hence $(H', \alpha \triangle \alpha')$, has a blocking pair;*

(h2) $\mathrm{ecycle}(H, \alpha \triangle \alpha') = \mathrm{ecycle}(H', \alpha \triangle \alpha')$ *is* 3*-connected;*

(h3) $\mathrm{ecycle}(G, \alpha \triangle \alpha') = \mathrm{ecycle}(G', \alpha \triangle \alpha')$ *is* 3*-connected.*

Lemma 37. *If Hypothesis 36 holds, then $(G, \alpha \triangle \alpha')$ has no intercepting pair.*

Proof of Lemma 16. We may assume by Lemma 24 that $\mathbb{F}'$ is the union of exactly two equivalence classes $\mathbb{F}_1$ and $\mathbb{F}_2$. Let $(G, \Sigma) \in \mathbb{F}_1$ and $(G', \Sigma') \in \mathbb{F}_2$. Again by Lemma 24, (G, Σ) and (G', Σ') are Ω-split siblings, i.e. they arise from a split template $\mathbb{T} = (H, \alpha, v, H', \alpha', v', \mathbb{S})$ and $H = G/\Omega$, $H' = G'/\Omega$. Because of Remark 20, we may assume that $\Sigma = \Sigma' = \alpha \triangle \alpha'$.

Theorem 27 implies that, $(G, \alpha \triangle \alpha')$ and $(G', \alpha \triangle \alpha')$ are either nova twins, or simple twins (after possibly replacing $(G, \alpha \triangle \alpha')$ and/or $(G', \alpha \triangle \alpha')$ by equivalent signed graphs). Suppose that $(G, \alpha \triangle \alpha')$ and $(G', \alpha \triangle \alpha')$ are simple twins. Then Remark 28 implies that $(G, \alpha \triangle \alpha')$ has a blocking pair. Hence, $(H, \alpha \triangle \alpha') = (G, \alpha \triangle \alpha')/\Omega \in \mathbb{F}$ has a blocking pair, a contradiction since by hypothesis, $\mathbb{F}$ has no blocking pairs. Thus, $(G, \alpha \triangle \alpha')$ and $(G', \alpha \triangle \alpha')$ must be nova twins.

Since $\mathbb{F}$ has no blocking pair, no signed graph equivalent to $(H, \alpha \triangle \alpha') \in \mathbb{F}$ has a blocking pair, i.e. (h1) of Hypothesis 36 holds. Since by hypothesis, $N = \mathrm{ecycle}(H, \alpha \triangle \alpha')$ and $M = \mathrm{ecycle}(G, \alpha \triangle \alpha')$ are 3-connected, conditions (h2) and (h3) hold as well. Hence, by Lemma 37, $(G, \alpha \triangle \alpha') \in \mathbb{F}_1$ has no intercepting pair, i.e. $\mathbb{F}_1$ has no intercepting pair. Similarly, we show that $\mathbb{F}_2$ has no intercepting pair either. $\square$

Thus it only remains to prove Lemma 37 which follows immediately from the next two results (Lemma 38 and Lemma 39).

Lemma 38. *Suppose Hypothesis 36 holds. If $(G, \alpha \triangle \alpha')$ has an intercepting pair, then some signed graph equivalent to $(G, \alpha \triangle \alpha')$ has a handcuff-separation.*

Proof. Suppose that, $(G, \alpha \triangle \alpha')$ has an intercepting pair (G_1, v_1) and (G_2, v_2). It follows from Remark 14 that we can find $\beta_1 \subseteq \delta_{G_1}(v_1) \cup \mathrm{loop}(G)$ and $\beta_2 \subseteq \delta_{G_2}(v_2) \cup \mathrm{loop}(G)$ such that $\beta_1 \triangle \beta_2$ is a signature of $(G, \alpha \triangle \alpha')$. As ecycle$(G, \alpha \triangle \alpha')$ is 3-connected by (h3) of Hypothesis 36, Proposition 25 implies that G_1 and G_2 are 2-connected (up to loops). Hence, $G_2 = \mathrm{W}_{\text{flip}}[G_1, \mathbb{R}]$ for some w-sequence $\mathbb{R}$ of G_1. It follows that $\mathbb{L} := (G_1, v_1, \beta_1, G_2, v_2, \beta_2, \mathbb{R})$ is a split template as it satisfies hypotheses (a), (b), (c) (see Section 3.2).

Lemma 30 implies that there exists a split-template $\mathbb{L}' := (G_1', v_1', \beta_1', G_2', v_2', \beta_2', \mathbb{R}')$ compatible with $\mathbb{L}$ which is simple or nova. Since $\mathbb{L}'$ is compatible with $\mathbb{L}$, both $\beta_1 \triangle \beta_1'$ and $\beta_2 \triangle \beta_2'$ are cuts of G_1, hence of G (as G and G_1 are equivalent they have the same cuts). It follows that $\beta_1' \triangle \beta_2'$ is a signature of $(G, \alpha \triangle \alpha')$. Hence, $(G_1', \beta_1' \triangle \beta_2')$ is equivalent to $(G, \alpha \triangle \alpha')$. Suppose that $\mathbb{L}'$ is simple. Then Remark 28 implies that $\{v_1', v_2'\}$ is a blocking pair of $(G_1', \beta_1' \triangle \beta_2')$. Then $(G_1', \beta_1' \triangle \beta_2')/\Omega$ has a blocking pair. But then Remark 10 implies that this signed graph is equivalent to $(H, \alpha \triangle \alpha')$, contradicting (h1). Hence, $\mathbb{L}'$ is nova. It follows in particular that $(G_1', \beta_1' \triangle \beta_2')$ must have a handcuff-separation. $\square$

Lemma 39. *Suppose Hypothesis 36 holds. Then* no *signed graph equivalent to $(G, \alpha \triangle \alpha')$ has a handcuff-separation.*

Hence, it only remains to prove Lemma 39.

We first need preliminaries. For a graph H, we say that $\mathbf{F} = (B_1, \ldots, B_t)$ is a *flower* if $B_1, \ldots, B_t$ is a partition of $E(H)$ and there exist distinct vertices $u_1, \ldots, u_t$ such that

(a) $t \geq 2$ and if $t = 2$ then $|B_1| > 1$ and $|B_2| > 1$,

(b) $H[B_i]$ is connected, for every $i \in [t]$, and

(c) $\mathscr{B}_H(B_i) = \{u_i, u_{i+1}\}$, for every $i \in [t]$ (where $t+1$ denotes 1).

For $i \in [t]$, B_i is a *petal* with *attachments* u_i and u_{i+1}.

The following technical result characterizes the possible 2-separations of graph G in Hypothesis 36.

Lemma 40. *Suppose Hypothesis 36 holds. Assume that $\mathbb{S} = \{X_1, \ldots, X_k\}$ is a w-star of H satisfying properties (a)-(c) of Section 4.2.2, with vertices $w_1, \ldots, w_k \in V(H) \cap V(G)$. Let v^+ and v^- denote the ends of Ω in G and let $X_0 := E(H) - (X_1 \cup \ldots \cup X_k)$. Then*

(1) $k = |\mathbb{S}| \geq 2$;

(2) for all $i \in [k]$, $\mathscr{B}_G(X_i) = \{v^-, v^+, w_i\}$;

(3) for all $i \in [k]$, $\big(G[X_i], (\alpha \triangle \alpha') \cap X_i\big) \setminus \delta_G(w_i)$ is bipartite;

(4) $\big(G[X_0], (\alpha \triangle \alpha') \cap X_0\big) \setminus \big[\delta_G(v^-) \cup \delta_G(v^+)\big]$ is bipartite.

Let Z be a 2-separation of G with $\mathscr{B}_G(Z) = \{z_1, z_2\}$ and $\Omega \notin Z$. Then, after possibly exchanging the labels of z_1 and z_2, one of the following holds:

(A) $Z \subseteq X_0$, $z_1 \in \{v^-, v^+\}$ and $z_2 \in V_G(X_0) - \{v^-, v^+\}$;

(B) There exists $i \in [k]$ such that $Z \subset X_i$, $z_1 = w_i$ and $z_2 \in V_G(X_i)$;

(C) There exist $i \in [k]$ and an edge $e = (w_i, z_2) \in X_0 \cap Z$ such that $Z \subset X_i \cup \{e\}$. Moreover, z_1 is a cut vertex of $G[X_i]$ separating $\{v^-, v^+\}$ and w_i;

(D) $\bar{Z} = \{\Omega, \Omega'\}$, where $\Omega' = (v^-, v^+)$ and $\Omega \notin \alpha \triangle \alpha'$, $\Omega' \in \alpha \triangle \alpha'$.

Figure 2 (left) illustrates properties (1)-(4) of the previous lemma (with $\alpha = \alpha_1$, $\alpha' = \alpha_2$, $v = v_1^-$, $v^+ = v_1^+$).

Proof of Lemma 40. Since $\mathbb{T}$ given in Hypothesis 36, is a template, $\alpha \subseteq \delta_H(v) \cup \mathrm{loop}(H)$. Hence,

$$\alpha \subseteq \delta_G(v^+) \cup \mathrm{loop}(H). \tag{$\star$}$$

Since $\mathbb{T}$ is a template, $\alpha' \subseteq \delta_{H'}(v')$. Edges of X_i incident to v' in H' are incident to w_i in G. Edges of X_0 incident to v' in H' are incident to v^- or v^+ in G. Hence,

$$\alpha' \subseteq \Big[\bigcup_{i \in [k]} X_i \cap \delta_G(w_i)\Big] \cup \Big(\big[\delta_G(v^-) \cup \delta_G(v^+)\big] \cap X_0\Big) \cup \mathrm{loop}(H). \tag{$\dagger$}$$

Note that, if $f \in \mathrm{loop}(H) \cap \alpha$, then f has ends v^- and v^+ in G. It follows from ($\star$), ($\dagger$) that $k \geq 2$, for otherwise $\mathbb{S} = \{X_1\}$ and $\mathscr{B}_H(X_1) = \{v, w_1\}$, and $\{v, w_1\}$ is a blocking pair of $(H, \alpha \triangle \alpha')$, contradicting (h1). Hence (1) holds. (2) holds by the definition of w-star. (3) and (4) also follow from ($\star$), ($\dagger$). By (h2) (resp. (h3)) and Proposition 25 we know that H (resp. G) is 2-connected, up to loops. We define $\Sigma := \alpha \triangle \alpha'$.

Claim 1. *If S is a 2-separation with $\mathscr{B}_G(S) = \{v^-, v^+\}$, then S or $\bar{S}$ is equal to $\{\Omega, \Omega'\}$ where $\Omega' = (v^-, v^+)$ and $\Omega' \in \Sigma$.*

Proof. One of $\mathscr{I}_G(S)$ or $\mathscr{I}_G(\bar{S})$ is empty, for otherwise v is a cut vertex of H separating $\mathscr{I}_G(S)$ from $\mathscr{I}_G(\bar{S})$, a contradiction as H is 2-connected, up to loops. After possibly replacing S by its complement, we have that $\mathscr{I}(S) = \emptyset$. By the definition of 2-separations, $|S| \geq 2$. Proposition 25 implies that no two edges with endpoints v^- and v^+, respectively, have the same parity. $\diamond$

Claim 2. *If $\{z_1, z_2\} = \{v^-, v^+\}$ then Z is of Type* (D).

Proof. It follows from Claim 1 and the fact that $\Omega \in \bar{Z}$. $\diamond$

Claim 3. *For all $i, j \in [k]$ ($i \neq j$) there exists a $w_i - w_j$ path in $G[E(G) \setminus (X_i \cup X_j)]$ avoiding v^- and v^+.*

Proof. If this is not the case, then v^-, v^+ is a 2-separation of G separating w_i and w_j. Hence, v is a cut vertex of H, a contradiction as H is 2-connected, up to loops. $\diamond$

Claim 4. *For any $i \in [k]$ and $z \in \{v^+, v^-, w_i\}$, $G[X_i] - z$ is connected.*

Proof. We consider the case where $z = w_i$ only as the other cases are similar. If $G[X_i] - w_i$ is not connected, it must have components $G[Z_1]$ and $G[Z_2]$. Since G is 2-connected, we may assume that $v^+ \in V_G(Z_1)$ and $v^- \in V_G(Z_2)$. Let $Z_1' := Z_1 \cup \{(w_i, u) \in E(G) : u \in V_G(Z_1)\}$. Note $\mathscr{B}_H(Z_1') = \{v, w_i\}$. Since z_1 is a blocking vertex of $(H[Z_1'], \Sigma \cap Z_1')$, Z_1' is not a handcuff-separation of (H, Σ). But this contradicts property (b) of nova templates (see Section 4.2.2). $\diamond$

Claim 5. *Suppose for some $i \in [k]$, $X_i \cap Z \neq \emptyset$ and $z_1, z_2 \in V_G(X_i)$. Then Z is of Type* (B) *or* (D).

Proof. Consider first the case where $Z \subseteq X_i$. If $w_i \notin \{z_1, z_2\}$ then (3) implies that $(G[Z], \Sigma \cap Z)$ is bipartite. Proposition 25 then implies that $|Z| = 1$, a contradiction. Hence, $z_j = w_i$ for some $j \in [w]$ and (B) holds. Thus we may assume that $Z_1, Z_2 \neq \emptyset$ where $Z_1 := Z \cap X_i$ and $Z_2 := Z \cap \bar{X}_i$.

Since $z_1, z_2 \in V_G(X_i)$, $\mathscr{B}_G(Z_2) \subseteq V_G(X_i)$. As G is 2-connected, up to loops, $|\mathscr{B}_G(Z_2) \cap \mathscr{B}_G(X_i)| \geq 2$. We may assume v^-, v^+ are not both included in $\mathscr{B}_G(Z_2)$, for otherwise, the edge $\Omega \in \bar{Z}$ implies that $\{z_1, z_2\} = \{v^-, v^+\}$, and hence, by Claim 2 that Z is of Type (D). Hence, (i) $\mathscr{B}_G(Z_2) = \{v^-, w_i\}$ or (ii) $\mathscr{B}_G(Z_2) = \{v^+, w_i\}$ and in particular, $Z_2 \subseteq X_0$. We assume that (ii) holds as the proof for (i) is the same.

Since v^+ is an end of $\Omega \in \bar{Z}$, and since $v^+ \in \mathscr{B}_G(Z_2)$, we may assume (after possibly exchanging the labels of z_1 and z_2) that $z_1 = v^+$. Because of Claim 2 we may assume that $z_2 \neq v^-$. Let $j \in [k]$ ($j \neq i$). Claim 4 implies that there exists a $v^- - w_j$ path P of $G[X_j] - v^+$. Claim 3 implies that there exists a $w_j - w_i$ path P' of $G[E(G) \setminus (X_i \cup X_j)]$ avoiding v^- and v^+. Then $Q := P \cup P'$ is a $v^- - w_i$ path of G. Note, that $Q \cap Z \subseteq Q \cap (X_i \cup Z_2) = \emptyset$. Since, $w_i \in \mathscr{B}_G(Z_2)$ and $Q \subseteq \bar{Z}$, we have $z_2 = w_i$.Hence, $\mathscr{B}_G(Z_2) = \{v^+, w_i\}$. It follows that $\mathscr{B}_G(Z_1) = \{v^+, w_i\}$. Since w_i is a blocking vertex of $(H[Z_1], \Sigma \cap Z_1)$, Z_1 is not a handcuff-separation of (H, Σ). But this contradicts property (b) of nova templates. $\diamond$

Claim 6. *If there exists $i \in [k]$ such that $Z \supseteq X_i$ then Z is of Type* (D).

Proof. As $\Omega \in \bar{Z}$, $\Omega = (v^-, v^+)$ and $\{v^-, v^+\} \in V_G(X_i)$. The result now follows from Claim 2. $\diamond$

Claim 7. *Suppose that for some $i \in [k]$, $Z \cap X_i, \bar{Z} \cap X_i \neq \emptyset$ and suppose that $z_2 \notin V_G(X_i)$. Then $z_1 \in \mathscr{I}_G(X_i)$ and z_1 is a cut vertex of $G[X_i]$ separating $\{v^-, v^+\}$ and w_i.*

Proof. Note that $z_1 \in V_G(X_i)$ for otherwise, $X_i \cup \{\Omega\} \subseteq \bar{Z}$, a contradiction. Suppose that $z_1 \in \mathscr{B}(X_i)$. Then Claim 4 implies that $G[X_i] - z_1$ is connected. It follows that $X_i - \delta_G(z_1)$ is included in Z or $\bar{Z}$. Hence, X_i is included in Z or $\bar{Z}$, a contradiction. Thus we may assume that $z_1 \in \mathscr{I}_G(X_i)$. Suppose, for a contradiction that z_1 is not a cut vertex of $G[X_i]$ separating w_i and $\{v^-, v^+\}$. Then there exists a $w_i - \{v^-, v^+\}$ path Q of $G[X_i] - z_1$. Because of Q and Ω, $\{w_i, v^-, v^+\} \subseteq \mathscr{I}_G(\bar{Z})$. Hence, every $z_1 - z_2$ path of $G[Z]$ intersects $\mathscr{I}_G(\bar{Z})$. It follows that $G[Z]$ is not connected, a contradiction as G is 2-connected, up to loops. $\diamond$

Claim 8. *There does not exists $i,j \in [k]$ ($i \neq j$) such that*

$$Z \cap X_i \neq \emptyset, \bar{Z} \cap X_i \neq \emptyset \qquad \textit{and} \qquad Z \cap X_j \neq \emptyset, \bar{Z} \cap X_j \neq \emptyset.$$

Proof. Suppose for a contradiction that the claim is not true. Consider first the case where $z_1, z_2 \in V(X_i)$. Then Claim 5 implies that Z is of Type (B) or (D). In all cases this implies that $Z \cap X_j = \emptyset$, a contradiction. Thus we may assume that $z_2 \notin V(X_i)$. Because of Claim 7, $z_1 \in \mathscr{I}_G(X_i)$ and z_1 is a cut vertex of $G[X_i]$ separating $\{v^-, v^+\}$ and w_i. In particular, $z_1 \notin V(X_j)$. Thus, again by Claim 7, $z_2 \in \mathscr{I}_G(X_j)$ and z_2 is a cut vertex of $G[X_j]$ separating $\{v^-, v^+\}$ and w_j. Since $\Omega = (v^+, v^-)$ and $\Omega \notin Z$, we have $v^+, v^- \in \mathscr{I}_G(\bar{Z})$ and $\{w_i, w_j\} \in \mathscr{I}_G(Z)$. Hence, there is no $\{v^-, v^+\} - \{w_i, w_j\}$ path of $G - (X_i \cup X_j)$. Suppose that there exists $B \subseteq E(G) - (X_1 \cup X_2)$ such that $V_G(B) \cap \{v^-, v^+\} \neq \emptyset$ and $G[B]$ is connected. Claim 1 implies that B consists of the edge $\Omega \notin \Sigma$ and possibly another edge parallel to Ω that is in Σ. It follows that, $\mathbb{S} = \{X_1, X_2\}$ and $i = 1, j = 2$. Finally, by Claim 3, there exists a $w_1 - w_2$ path in $G[E(G) \setminus (X_1 \cup X_2)]$ avoiding v^- and v^+. Hence, G has a flower,

$$\mathbf{F} = (B \cup (X_1 \cup X_2) \cap \bar{Z}, X_1 \cap Z, X_0 - B, X_2 \cap Z\}).$$

Let $\mathbb{L}$ denote the w-sequence $\big((X_1 \cap Z) \cup (X_0 - B), X_1 \cap Z\big)$ of G and let $\hat{G} := \mathrm{W}_{\text{flip}}(G, \mathbb{L})$. It can be readily checked, see ($\star$), ($\dagger$), that $(\hat{G}, \Sigma)$ has a blocking pair. Hence, so does $(\hat{G}, \Sigma)/\Omega$. Remark 10 implies that this signed graph is equivalent to (H, Σ), contradicting (h1). $\diamond$

We may assume outcome (D) does not occur. It follows from Claim 6 that, for all $i \in [k]$, $X_i \cap \bar{Z} \neq \emptyset$. It follows now, from Claim 8, that one of the following holds,

(T1) $Z \subseteq X_0$;

(T2) $\exists i \in [k]$ such that $Z \subset X_i$;

(T3) $\exists i \in [k]$ such that $Z \subset X_i \cup X_0$, $\bar{Z} \cap X_i \neq \emptyset$, $Z \cap X_0 \neq \emptyset$.

Suppose (T1) holds. Then $z_1, z_2 \in V_G(X_0)$. If $\{z_1, z_2\} \cap \{v^-, v^+\} = \emptyset$ then (4) implies that $(G[Z], \Sigma \cap Z)$ is bipartite. Proposition 25 then implies that $|Z| = 1$, a contradiction. Thus we may assume that $z_1 \in \{v^-, v^+\}$ and by Claim 2 that $z_2 \notin \{v^-, v^+\}$. Hence, (A) holds. Suppose (T2) holds. Then $z_1, z_2 \in V_G(X_i)$ and by Claim 5, Z is of Type (B), or (D). Suppose (T3) holds. If $z_1, z_2 \in V_G(X_i)$, then Claim 5 implies that Z if of Type (B), or (D). Otherwise we may assume that $z_2 \notin V_G(X_i)$. By Claim 7, we may assume that $z_1 \in \mathscr{I}_G(X_i)$ and z_1 is a cut vertex of $G[X_i]$ separating $\{v^-, v^+\}$ and w_i. It follows that $Z \cap X_0$ is a 2-separation with $\mathscr{B}(Z \cap X_0) = \{w_i, z_2\}$. Because of (4), $\big(G[Z \cap X_0], \Sigma \cap (Z \cap X_0)\big)$ is bipartite. Proposition 25 then implies that $Z \cap X_0$ consists of a single edge $e = (w_i, z_2)$ and (C) holds. $\square$

Let (G, Σ) be a signed graph. We say that a 2-separation X of G is a *bracelet separation* if $(G[X], \Sigma \cap X)$ is not bipartite, and some $v \in \mathscr{B}(X)$ is a blocking vertex of $(G[X], X \cap \Sigma)$. A flower is *maximal* if no petal has a cut-vertex separating its attachments. A petal of a flower that is an edge is a *petal edge*.

We say that a flower **F** is a *Type 1* flower of (G,Σ) if

(a) **F** is a maximal flower;

(b) **F** has exactly two petals;

(c) one petal is a bracelet separation of (G,Σ).

We say that a flower **F** is a *Type 2* flower of (G,Σ) if

(a) **F** is a maximal flower;

(b) **F** has exactly three petals;

(c) one petal is a bracelet separation of (G,Σ) and one petal is a petal edge.

Lemma 41. *Suppose Hypothesis 36 holds. Then every maximal flower of* $(G,\alpha\triangle\alpha')$ *is of Type 1 or Type 2.*

Proof. We say that a 2-separation Z of G, where $\Omega \notin Z$, is of Type (A), (B), (C), or (D), respectively, if outcome (A), (B), (C), or (D), respectively, occurs when we apply Lemma 40 to the separation Z. The following result is a direct consequence of part (3) and (4) of Lemma 40.

Claim. *Let Z be a* 2*-separation of* $(G,\alpha\triangle\alpha')$ *where* $\Omega \notin Z$*.*

(1) If Z if of Type (A) *or* (B) *then Z is a bracelet separation;*

(2) If Z if of Type (C) *then Z is obtained, by adding in a flower, a petal edge to a bracelet separation;*

(3) If Z is of Type (D) *then $\bar{Z}$ is a bracelet separation.*

Let $\mathbf{F} = (Z_0,\dots,Z_t)$be an arbitrary maximal flower of $(G,\alpha\triangle\alpha')$. We may assume that $\Omega \in Z_0$. No two of $Z_1,\dots,Z_t$ are petal edges, for otherwise these petal edges are in series contradicting (h3) of Hypothesis 36. Hence, at most one of $Z_1,\dots,Z_t$ is a petal edge. Suppose first that all but one of $Z_1,\dots,Z_t$ is a petal edge. Then either $t=1$ and Z_1 is not a petal edge, or $t=2$ and exactly one of Z_1 and Z_2 (say Z_2) is a petal edge. In the case $t=1$, either Z_1 is of Type (D), and by the Claim, $Z_0=\bar{Z}_1$ is a bracelet separation, or Z_1 is not of Type (D) and by the Claim, Z_1 is a bracelet separation. Either way, **F** is a flower of Type 1. In the case $t=2$, Z_1 is not of Type (D) (as $\mathscr{I}_G(\bar{Z}_1)\neq\emptyset$) and by the Claim Z_1 is a bracelet separation. Then **F** is a flower of Type 2.

Thus we may assume that, for some $i,j\in[t]$ where $i<j$, $|Z_i|>1$ and $|Z_j|>1$. Let $Z := Z_1\cup\ldots\cup Z_i$ and $Z' := Z_{i+1}\cup\ldots\cup Z_t$. Since $\mathscr{I}_G(\bar{Z}),\mathscr{I}_G(\bar{Z}')\neq\emptyset$, Z,Z' are not of Type (D). Hence, by the Claim, Z,Z' are bracelet separations. Observe that if $Z\cup Z'$ is of Type (D), then some signed graph equivalent to $(G, \alpha \triangle \alpha')$ has a blocking pair, contradicting (h1). Thus we may assume that $Z \cup Z'$ is not of Type (D). It follows by the Claim that $Z \cup Z'$ is a bracelet separation. However, it is not possible that all of $Z, Z', Z \cup Z'$ are bracelet separations, a contradiction. □

Lemma 41 implies that the signed graph $(G, \alpha \triangle \alpha')$ in Hypothesis 36 has no handcuff-separations. To prove Lemma 39 however, we need to show that no signed graphs *equivalent* to $(G, \alpha \triangle \alpha')$ has a handcuff-separation. In order to do this we need to be able to relate 2-separations and flowers between equivalent graphs and signed graphs.

Lemma 42. *Let $\hat{G}$ be a 2-connected graph and let X be a 2-separation of $\hat{G}$. Let G be equivalent to $\hat{G}$. Then there exists a maximal flower $\mathbf{F} = (B_1, \ldots, B_t)$ of G, such that X is equal to the union of a subset of the petals of G. In particular, if every maximal flower of G has at most three petals, then every 2-separation of $\hat{G}$ is a 2-separation of G.*

Proof. Given a graph H, denote by $c(H)$ the number of components of H. Let $M := \text{cycle}(G) = \text{cycle}(\hat{G})$. Then for any $Y \subseteq E(\hat{G})$, the connectivity function $\lambda_M(Y)$ satisfies the relation,

$$\lambda_M(Y) = |V_G(Y) \cap V_G(\bar{Y})| - c(G[Y]) - c(G[\bar{Y}]) + 2.$$

Since X is a 2-separation of $\hat{G}$, $\lambda_M(X) = 2$. Hence,

$$|V_G(X) \cap V_G(\bar{X})| = c(G[X]) + c(G[\bar{X}]).$$

Construct an auxiliary graph H as follows: the vertices of H correspond to the components of $G[X]$ and to the components of $G[\bar{X}]$. Join two vertices of H by k parallel edges if the corresponding components have k vertices in common. Then $|V(H)| = |E(H)|$. Moreover, since $\hat{G}$ (and therefore G) is 2-connected, $deg_H(v) \geq 2$ for every $v \in V(H)$ and H is connected. It follows that H consists of a circuit and the result follows. □

The signed graph $\mathscr{F}_7$ is obtained by replacing every edge in a triangle by two parallel edges, one odd, one even, and by adding an odd loop (see Figure 3). Note that ecycle$(\mathscr{F}_7)$ is the matroid F_7.

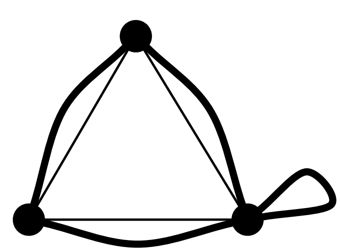

Figure 3: The signed graph $\mathscr{F}_7$.

Lemma 43. *Let (G,Σ) be a signed graph and suppose that no signed graph equivalent to (G,Σ) has a blocking pair. Let $\mathbf{F}$ be a Type 1 or Type 2 flower of (G,Σ) and let Z be a petal of $\mathbf{F}$. Let $(\hat{G},\Sigma)$ be a signed graph equivalent to (G,Σ). Then Z is is not a handcuff-separation of $(\hat{G},\Sigma)$.*

Proof. Suppose for a contradiction that Z is a handcuff-separation of $(\hat{G},\Sigma)$. We will only prove the case for the Type 1 flowers as the proof for Type 2 is similar. Then $\mathbf{F}=\{Z,Z'\}$ and one of Z,Z' is a bracelet separation. Observe that $\mathbf{F}$ is a flower of $\hat{G}$. Since $|\mathbf{F}|\leq 3$, we may assume that there exists a w-sequence $\mathbb{S}$ of G where $\hat{G}=\mathrm{W}_{\text{flip}}[G,\mathbb{S}]$ and, for all $S\in\mathbb{S}$, we have $S\subseteq Z$.

Consider first the case where Z' is a bracelet separation of (G,Σ). Then Z' remains a bracelet separation of $(\hat{G},\Sigma)$. But then, as Z is a handcuff-separation, $\mathscr{B}(Z)$ is a blocking pair, contradicting our hypothesis. Consider now the case where Z is a bracelet separation of (G,Σ). Construct a signed graph (H,Γ) from $(G[Z],\Sigma\cap Z)$ by adding two parallel edges, one odd, one even, between vertices of $\mathscr{B}_G(Z)$ and by adding an odd loop. Let $\hat{H}:=\mathrm{W}_{\text{flip}}[H,\mathbb{S}]$. Note that (H,Γ) and $(\hat{H},\Gamma)$ are equivalent hence they are representations of the same even cycle matroid. To obtain a contradiction we will show that $\text{ecycle}(H,\Gamma)$ is a graphic matroid but that $\text{ecycle}(\hat{H},\Gamma)$ is not graphic. Since Z is a bracelet separation of (G,Σ), (H,Γ) has a blocking vertex. It follows from Remark 6 that (H,Γ) is graphic. Since Z is a handcuff-separation of $(\hat{G},\Sigma)$, $(\hat{H},\Gamma)$ contains the signed graph $\mathscr{F}_7$ as a minor. As $\text{ecycle}(\mathscr{F}_7)$ is the matroid F_7, it follows in particular that $\text{ecycle}(\hat{H},\Gamma)$ is not graphic. □

We are now ready for the proof of the last remaining lemma.

Proof of Lemma 39. Let $(\hat{G},\alpha\triangle\alpha')$ be a signed graph equivalent to $(G,\alpha\triangle\alpha')$. Let Z be an arbitrary 2-separation of $\hat{G}$. It follows from the fact that maximal flowers of G have at most three petals (Lemma 41) and Lemma 42 that Z is a 2-separation of G. Lemma 41 implies that Z is a petal of a flower of Type 1 or Type 2 of $(G,\alpha\triangle\alpha')$. It follows from Lemma 43 that Z is not a handcuff-separation of $(\hat{G},\alpha\triangle\alpha')$. □

References

[1] T.H. Brylawski, *A decomposition for combinatorial geometries*, Trans. Amer. Math. Soc. **171** (1972), 235–282.

[2] J. Geelen. B. Gerards, G. Whittle, *Towards a matroid-minor structure theory*.

[3] B. Gerards, *A few comments on isomorphism of even cycle spaces*, Unpublished manuscript.

[4] B. Guenin, I. Pivotto, P. Wollan, *Relation between pairs of representations of signed binary matroids*, accepted by the SIAM J. Discrete Math.

[5] I. Pivotto, *Even cycle and even cut matroids*, Ph.D thesis, University of Waterloo, (2011).

[6] J. Oxley, *Matroid Theory*, Oxford University Press, (1992).

[7] Robertson N., Seymour P.D, *Graph Minors. XX. Wagner's conjecture*, J. of Comb. Theory, Series B 92 (2), pages 32535, (2004).

[8] P.D. Seymour, *A note on the production of matroid minors*, J. Combin. Theory Ser. B **22** (1977), 289–295.

[9] P.D. Seymour, *Decomposition of Regular Matroids*, J. Combin. Theory Ser. B **28** (1980), 305–359.

[10] D. Slilaty, *Bias matroids with unique graphical representations*, Discrete Math. **306** (2006), 1253–1256.

[11] W. T. Tutte, *An algorithm for determining whether a given binary matroid is graphic*, Proc. Amer. Math. Soc. 11: 905–917, (1960).

[12] Tutte W.T. *Matroids and graphs*, Trans. Amer. Math. Soc. 90, pages 527–552, (1959).

[13] H. Whitney, *2-isomorphic graphs*, Amer. J. Math. **55** (1933), 245–254.