# Eigenvalues and Search Engines

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These notes complement Kleinberg's paper by giving a full proof of the convergence of the method.

## 1 The convergence of Hits

Let $A$ be the adjacency matrix of the hubs/authorities graph, and let

$$B := A^T A$$

**Exercise 1** *Show that $B$ is symmetric.*

Starting from a unit vector $\hat{a}_0$ the Hits algorithm computes the normalized sequences

$$\hat{h}_k = \frac{A\hat{a}_{k-1}}{||A\hat{a}_{k-1}||}.$$

and

$$\hat{a}_k = \frac{A^T\hat{h}_k}{||A^T\hat{h}_k||}$$

so that

$$\hat{a}_k = \frac{B\hat{a}_{k-1}}{||B\hat{a}_{k-1}||}$$

**Exercise 2** *Prove or disprove: $AA^T = A^T A$.*

We want to show that, unless we are very unlucky with our choice of $\hat{a}_0$, the sequence $\hat{a}_k$ *always* converge (rather quickly) to the principal eigenvector of $B$ of unit norm.

We will work with the the $n$ eigenvalues of $B$ and consider them as an *ordered* sequence

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

where

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$$

and will denote the corresponding eigenvectors as

$$b_1, b_2, \ldots, b_n.$$

These eigenvectors can be assumed to be of *unit norm*.

**Exercise 3** *Show that if $x$ is an eigenvector of a matrix $A$ then so is $cx$, for any $c$.*

**Exercise 4** *Show that eigenvectors corresponding to different eigenvalues are linearly independent.*

**Exercise 5** *Show that eigenvectors corresponding to different eigenvalues of a symmetric matrix are orthonormal.*

**Exercise 6** *Show that the eigenvalues of a symmetric matrix are real.*

In the rest of the section we shall prove the following.

**Theorem 1** *If the eigenvalues of $B$ are all different, then $\hat{a}_k$ converges to $b_1$, provided that $\hat{a}_0$ and $b_1$ are not orthogonal.*

In the next section we will show the same convergence under the weaker assumption $|\lambda_1| > |\lambda_2|$.

Consider the non normalized sequence

$$a_k = Ba_{k-1} = B^k a_0.$$

where $a_0 = \hat{a}_0$. By Exercise 4 the $n$ eigenvalues form a basis. Thus,

$$a_0 = \sum_{i=1}^{n} c_i b_i$$

for some vector $c := (c_1, \ldots, c_n)$. Let us start by unfolding $a_1$,

$$a_1 = Ba_0 = B\sum_{i=1}^{n} c_i b_i = \sum_{i=1}^{n} c_i B b_i = \sum_{i=1}^{n} c_i \lambda_i b_i$$

and then $a_2$,

$$a_2 = Ba_1 = B\sum_{i=1}^{n} c_i \lambda_i b_i = \sum_{i=1}^{n} c_i \lambda_i B b_i = \sum_{i=1}^{n} c_i \lambda_i^2 b_i$$

Thus in general, by a trivial induction,

$$a_k = \sum_{i=1}^{n} c_i \lambda_i^k b_i.$$

This vector tends to a vector parallel to $b_1$. To see this, consider the vector

$$v_k := \frac{a_k}{(\lambda_1)^k} = c_1 b_1 + \sum_{i=2}^{n} c_i \left(\frac{\lambda_i}{\lambda_1}\right)^k b_i.$$

By our assumption that $a_0$ is not perpendicular to $b_1$, $c_1 \neq 0$. Therefore $v_k$ tends to $c_1 b_1$ when $k$ goes to infinity.

**Note:** the above is nothing else than the well-known power method for computing the eigenvalues of a matrix.

Thus the non normalized sequence converges to a vector parallel to $b_1$. Let us show now that the normalized sequence converges to $b_1$. The normalized sequence is defined as

$$\begin{aligned} \hat{a}_0 &= a_0 \\ \hat{a}_k &= \frac{B\hat{a}_{k-1}}{||B\hat{a}_{k-1}||} \end{aligned}$$

We show by induction that

$$\hat{a}_k = \frac{a_k}{||a_k||}$$

i.e., $\hat{a}_k$ is a unit vector parallel to $a_k$. The basis holds trivially. For the inductive step,

$$\hat{a}_k = \frac{B\hat{a}_{k-1}}{||B\hat{a}_{k-1}||} = \frac{Ba_{k-1}}{||a_{k-1}||\,||B\hat{a}_{k-1}||} = \frac{Ba_{k-1}\,||a_{k-1}||}{||a_{k-1}||\,||Ba_{k-1}||} = \frac{Ba_{k-1}}{||Ba_{k-1}||} = \frac{a_k}{||a_k||}.$$

## 2 Extending the method

We now show a more general form of Theorem 1, namely,

**Theorem 2** *If $|\lambda_1| > |\lambda_2|$ then $a_k$ converges to $b_1$, provided that $a_0$ and $b_1$ are not orthogonal.*

This result too is well-known in computational linear algebra. To establish the result we will show that under the current hypothesis we can find a set of eigenvectors that form an orthonormal basis. Then the result follows from the analysis of the previous section.

We will use a well-known result of linear algebra that says that *any* square matrix can be decomposed as the product of an upper-triangular matrix with a unitary matrix and its inverse.

**Definition 1** *A matrix $U$ is* unitary *if (a) its columns are orthogonal, unit vectors and (b) $U^T = U^{-1}$, i.e, $U^TU = I$.*

**Exercise 7** *Show that the product of unitary matrices is unitary.*

**Exercise 8** *Let $A$ and $B$ be square matrices. Show that $(AB)^T = B^TA^T$. What if the matrices are not square?*

**Lemma 1 (Schur's Normal Form)** *Any square matrix $A$ can be decomposed as*

$$A = UTU^T$$

*where $U$ is unitary and $T$ is upper-triangular.*

**Proof.** The proof is by induction on $n$, the number of rows of $A$. The base case ($n = 1$) is trivial. For $n > 1$, let $x_1$ be the unit-norm eigenvector corresponding to $\lambda_1$ (every non null matrix has at least one eigenvalue). Let $y_2, \ldots, y_n$ be unit vectors such that $x_1, y_2, \ldots, y_n$ form a basis. The matrix

$$U_1 := [x_1|y_2|\ldots|y_n]$$

is unitary. Consider the matrix

$$B := U_1^TAU_1.$$

This matrix is of the form

$$B = \begin{bmatrix} \lambda_1 & v_1 \\ z & A_1 \end{bmatrix}$$

where $z$ is a vector of $n-1$ 0's, $v_1$ is some real vector with $n-1$ components, and $A_1$ is a $n-1 \times n-1$ real matrix. We now apply the induction hypothesis to the matrix $A_1$ obtaining

$$A_1 = UTU^T$$

where $U$ is unitary and $T$ upper triangular. If we define

$$U_2 := \begin{bmatrix} 1 & z^T \\ z & U \end{bmatrix}$$

where $z$ is a vector of $n-1$ zeroes, then $U_2$ is unitary. Now,

$$A = U_1 B U_1^T = U_1 U_2 \begin{bmatrix} \lambda_1 & v_1' \\ 0 & T \end{bmatrix} U_2^T U_1^T.$$

The claim follows from Ex. 7. ⊛

**Theorem 3** *Let $A$ be a symmetric matrix. Then we can find $n$ orthogonal eigenvectors of unit norm.*

**Proof.** By Lemma 1, $A = UTU^T$, but $A$ is symmetric hence,

$$UTU^T = A = A^T = (UTU^T)^T = UT^TU^T.$$

Thus $T = T^T$, but this implies that $T$ is a diagonal matrix. Since $U$ is unitary, $T$ then has the $n$ eigenvalues of $A$ along its diagonal. ⊛