On the $L(h,k)$-Labeling of Co-Comparability Graphs ⋆

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Abstract. Given two non negative integers h and k, an $L(h,k)$-*labeling* of a graph $G=(V,E)$ is a map from V to a set of labels such that adjacent vertices receive labels at least h apart, while vertices at distance at most 2 receive labels at least k apart. The goal of the $L(h,k)$-labeling problem is to produce a legal labeling that minimizes the largest label used. Since the decision version of the $L(h,k)$-labeling problem is NP-complete, it is important to investigate classes of graphs for which the problem can be solved efficiently.
Along this line of though, in this paper we deal with co-comparability graphs and two of its subclasses: interval graphs and unit-interval graphs. Specifically, we provide, in a constructive way, the first upper bounds on the $L(h,k)$-number of co-comparability graphs and interval graphs. To the best of our knowledge, ours is the first reported result concerning the $L(h,k)$-labeling of co-comparability graphs.
In the special case where $k=1$, our result improves on the best previously-known approximation ratio for interval graphs.

Keywords: $L(h,k)$-Labeling, co-comparability graphs, interval graphs, unit-interval graphs.

1 Introduction

Graph coloring is, without doubt, one of the most fertile and widely studied areas in graph theory, as evidenced by the list of solved and unsolved problems in Jensen and Toft's comprehensive book on graph coloring [21]. The classic

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problem of (vertex) coloring, asks for an assignment of non-negative integers (colors) to the vertices of a graph in such a way that adjacent vertices receive distinct colors. Of interest, of course, are assignments (colorings) that minimize the number of colors used.

In this paper we focus on a generalization of the classic vertex coloring problem – the so-called *$L(h,k)$-labeling problem* – that asks for the smallest λ for which it is possible to assign integer labels $\{0,\dots,\lambda\}$ to the vertices of a graph in such a way that vertices at distance at most two receive colors at least k apart, while adjacent vertices receive labels at least h apart. The *span* of a $L(h,k)$-labeling is the difference between the maximum and the minimum label used. In the remainder of this work we shall follow established practice and refer to the largest label in an optimal $L(h,k)$-labeling for graph G as $\lambda_{h,k}(G)$.

We note that for $k=0$, the $L(h,k)$-labeling problem coincides with the usual vertex coloring; for $h=k$, we obtain the well-known 2-distance coloring, which is equivalent to the vertex coloring of the square of a graph.

The $L(h,k)$-labeling problem arises in many applications, including the design of wireless communication systems [20], radio channel assignment [8, 26], data distribution in multiprocessor parallel memory systems [4, **?**], and scalability of optical networks [1, 36], among many others.

The decision version of the vertex coloring problem is NP-complete in general [16], and it remains so for most of its variations and generalizations. In particular, it has been shown that the decision version of the $L(h,k)$-labeling problem is NP-complete even for $h=k=1$ [20, 25]. Therefore, the problem has been widely studied for many particular classes of graphs. For a survey of recent results we refer the interested reader to [9].

In this paper we deal with co-comparability graphs and two of its subclasses: interval graphs and unit-interval graphs. The literature contains a plethora of papers describing applications of these graphs to such diverse areas as archaeology, biology, psychology, management and many others (see [17, **?**,27, 29, 30]).

In the light of their relevance to practical problems, it is somewhat surprising to note the dearth of results pertaining to the $L(h,k)$-labeling of these graph classes. For example, a fairly involved web search has only turned up no results on the $L(h,k)$-labeling of co-comparability graphs and, as listed below, only two results on the $L(h,k)$-labeling of interval graphs and unit-interval graphs.

- In [33] the special case $h=2$ and $k=1$ is studied; the author proves that $2\chi(G)-2 \leq \lambda_{2,1}(G) \leq 2\chi(G)$ for unit interval graphs, where $\chi(G)$ is the chromatic number of G. In terms of the maximum degree Δ, as $\chi(G) \leq \Delta+1$, the upper bound becomes $\lambda_{2,1}(G) \leq 2(\Delta+1)$, and this value is very close to be tight, as the clique K_n, that is an interval graph, has $\lambda_{2,1}(K_n) = 2(n-1) = 2\Delta$.
- In [3] the authors present a 3-approximate algorithm for $L(h,1)$-labeling interval graphs and show that the same approximation ratio holds for the $L(h,k)$-labeling problem of unit-interval graphs.

One of our main contributions is to provide, in a constructive way, the first upper bounds on the $L(h, k)$-number of co-comparability and interval graphs. In the special case where $k = 1$, our result improves on the best previously-known approximation ratio for interval graphs.

This remainder of the paper is organized as follows: Section 2 is devoted to definitions and a review of preliminary results; in particular we show that the $L(1, 1)$-labeling problem is polynomially solvable for the three classes of graphs discussed in this work. Sections 3 and 4 focus, respectively, on the $L(h, k)$-labeling problem on co-comparability and interval graphs. Finally, Section 5 offers concluding remarks and open problems.

2 Preliminaries

The graphs in the work are simple, with no self-loops or multiple edges. We follow standard graph-theoretic terminology compatible with [17] and [5].

Fig. 1. *Illustrating two forbidden configurations.*

Vertex orderings have proved to be useful tools for studying structural and algorithmic properties of various graph classes. For example, Rose, Tarjan and Lueker [32] and Tarjan and Yannakakis [35] have used the well known simplicial ordering of the vertices of a chordal graph to obtain simple recognition and optimization algorithms for this class of graphs. To make this work as self-contained as possible, suffice it to say that a graph $G = (V, E)$ has a *simplicial ordering* if its vertices can be enumerated as $v_1, v_2, \ldots, v_n$ in such a way that for all subscripts $i,\ j,\ k$, with $1 \leq i < j < k \leq n$, the presence of the edges v_iv_k and v_jv_k implies the existence of the edge v_iv_j. Refer to Figure 1(a) for a forbidden configuration for a simplicial order.

Figure 1(b) illustrates a broader forbidden configuration that we shall refer to as the *umbrella*. Kratsch and Stewart [22] have shown that a graph is a *co-comparability graph* if and only if its vertices can be enumerated as $v_1, v_2, \ldots, v_n$ in such a way that for all subscripts $i,\ j,\ k$, with $1 \leq i < j < k \leq n$, the presence of the edges v_iv_k implies the presence of at least one of the edges and v_iv_k or v_iv_j. For alternate definitions of co-comparability graphs we refer to [19].

Later, Olariu [28] proved that a graph is an *interval graph* if and only if its vertices can be ordered as $v_1, v_2, \ldots, v_n$ in such a way that for all subscripts

$i,\ j,\ k$, with $1 \leq i < j < k \leq n$, the presence of the edge $v_i v_k$ implies the presence of the edge $v_i v_j$.

Finally, Looges and Olariu [24] showed that a graph is a *unit interval graph* if its vertices can be ordered as $v_1, v_2, \ldots, v_n$ in such a way that for all subscripts $i,\ j,\ k$, with $1 \leq i < j < k \leq n$, the presence of the edge $v_i v_k$ implies the presence of the edges $v_i v_j$ and $v_j v_k$.

The next proposition summarizes of previous discussion.

Proposition 1. *Let $G = (V, E)$ be a graph.*

1. *G is a co-comparability graph if and only if there exists an ordering of its vertices $v_0 < \ldots < v_{n-1}$ such that if $v_i < v_j < v_l$ and $(v_i, v_l) \in E$ then either $(v_i, v_j) \in E$ or $(v_j, v_l) \in E$ [22];*
2. *G is an interval graph if and only if there exists an ordering of its vertices $v_0 < \ldots < v_{n-1}$ such that if $v_i < v_j < v_l$ and $(v_i, v_l) \in E$ then $(v_i, v_j) \in E$ [28];*
3. *G is a unit-interval graph if and only there exists an ordering of its vertices $v_0 < \ldots < v_{n-1}$ such that if $v_i < v_j < v_l$ and $(v_i, v_l) \in E$ then $(v_i, v_j) \in E$ and $(v_j, v_l) \in E$ [24].*

In the remainder of this work we shall refer to a linear orders satisfying the above proposition as *canonical* and to the property that characterizes which edges must exist in a certain class as the *umbrella property* of that class. Also, Figure 2 summarizes the umbrella properties for co-comparability, interval, and unit-interval graphs. Observe that Proposition 1 confirms the well-known fact that unit-interval graphs ⊆ interval graphs ⊆ co-comparability graphs.

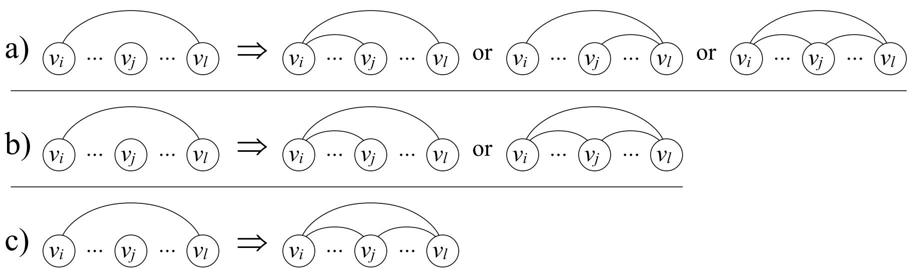

Fig. 2. *Illustrating the umbrella properties for a) co-comparability, b) interval, and c) unit-interval graphs.*

Before proving general results concerning the $L(h, k)$-labeling of the above classes of graphs, we make a few observations about the corresponding $L(1, 1)$-labelings. To begin, we observe that unit-interval, interval and co-comparability graphs are all perfect graphs and hence the vertex-coloring problem is polynomially solvable [17]. As already mentioned, the $L(1, 1)$-labeling problem for a graph G is exactly the vertex-coloring problem for its square graph G^2 (i.e. the graph having the same vertex set as G and having an edge connecting u to v if and only if u and v are at distance at most 2 in G). Since all these classes are closed under powers [14, 31], the following theorem holds:

Theorem 1. *The $L(1,1)$-labeling problem is polynomially solvable for unit-interval, interval and co-comparability graphs.*

3 The $L(h,k)$-Labeling of Co-Comparability Graphs

Given a co-comparability graph $G = (V,E)$ of maximum degree Δ, in view of the umbrella property (Proposition 1 item 1), if $(v_i, v_l) \in E$ and $v_i < v_l$ then all the $l-i-1$ vertices between v_i and v_l must be connected with at least one of these two vertices: d' are connected to v_i and d'' are connected to v_l, with $l-i-1 \leq d' + d''$.

The degree, $d(v_i)$, of v_i is at least $d'+1$, analogously $d(v_l) \geq d''+1$. Since the maximum degree is Δ we have $2\Delta \geq d' + d'' + 2 \geq l - i + 1$. Let us formalize this fact in the following proposition:

Proposition 2. *Given a co-comparability graph of maximum degree Δ, if $(v_i, v_l) \in E$ and $v_i < v_l$ then $l-i < 2\Delta$; if v_i and v_l are at distance 2 and $v_i < v_l$ then $l - i < 4\Delta$.*

Lemma 1. *A co-comparability graph G can be $L(h,k)$-labeled with span at most $2\Delta h + k$ if $k \leq \frac{h}{2}$.*

Proof. Let us consider the following ordered set of labels: $0, h, 2h, \ldots, 2\Delta h, k, h+k, 2h+k, \ldots, 2\Delta h + k$.

Let us label all vertices of G with labels in the given order following a canonical order of G's vertices; once the labels have been finished, we start again from label 0.

We will now prove that such a labeling is a feasible $L(h,k)$-labeling by showing that adjacent vertices are labeled with colors at least h apart and that vertices at distance 2 are labeled with colors at least k apart. The proofs are by contradiction and v_i and v_l are any two vertices with $i < l$.

Distance 1. Let v_i and v_l be adjacent vertices, assume by contradiction that their labels $l(v_i)$ and $l(v_l)$ differ by less than h. Then only two cases are possible:

(1.1) $l(v_i) = sh$, for some s such that $0 \leq s \leq 2\Delta$. Then $l(v_l) - l(v_i)$ can be smaller than h only if either $l(v_l) = sh + k$ or $l(v_l) = (s-1)h + k$. In both cases $l - i \geq (2\Delta - s) + (s-1) + 1 = 2\Delta$ as illustrated in Figure 3. This is impossible, for otherwise either v_i or v_l would have degree greater than Δ (see Proposition 2.) Notice that $l(v_l)$ cannot be sh because there are 4Δ distinct labels and $l - i$ is bounded by 2Δ.

(1.2) $l(v_i) = sh+k$, with $0 \leq s \leq 2\Delta$. Then $l(v_l)$ must be either sh or $(s+1)h$. In both cases $l-i \geq (2\Delta-s-1)+s+1 = 2\Delta$. Again, this is impossible. As in the previous case, $l(v_l)$ cannot be equal to $l(v_i)$.

Distance 2. Let v_i and v_l be at distance two with labels $l(v_i)$ and $l(v_l)$ that differ by less than k. Since $k \leq \frac{h}{2}$ it must be $l(v_i) = l(v_l)$, and therefore that $l - i = 4\Delta + 2$, i.e. the number of the different labels. This contradicts Proposition 2.

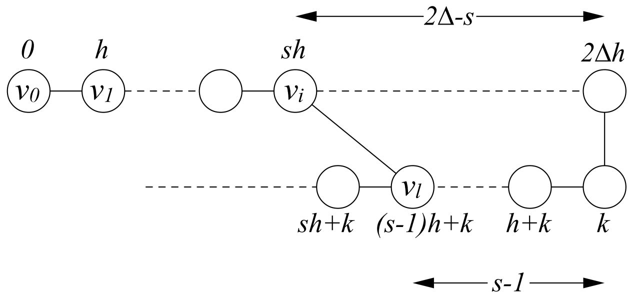

Fig. 3. Scheme for labeling vertices of a co-comparability graph.

Lemma 2. *A co-comparability graph G can be $L(h,k)$-labeled with span at most $4k\Delta + k$, if $k \geq \frac{h}{2}$.*

Proof. The proof is analogous to the one of Lemma 1. The only difference is the ordered set of labels used: $0, 2k, 4k, \ldots, 4k\Delta, k, 3k, 5k, \ldots, 4k\Delta + k$.

We can summarize both previous results in the following theorem:

Theorem 2. *A co-comparability graph G can be $L(h,k)$-labeled with span at most $2\Delta \max\{h, 2k\} + k$.*

4 The $L(h,k)$-Labeling of Interval Graphs

If the graph G is an interval graph, we can exploit its particular umbrella property to derive better bounds on $\lambda_{h,k}(G)$.

First observe that the degree of any vertex v_i connected to a vertex v_l, $v_i < v_l$, is at least $l - i$; furthermore, if $i \neq 0$ then the degree of v_i is at least $l - i + 1$ if G is connected, because at least one edge must reach v_i from vertices preceding it in the ordering.

Proposition 3. *Given a connected interval graph of maximum degree Δ, if $(v_i, v_l) \in E$ and $v_i < v_l$ then $l - i \leq \Delta$ and, if $i \neq 0$ then $l - i < \Delta$; if v_i and v_l are at distance 2 and $v_i < v_l$ then $l - i \leq 2\Delta - 1$.*

Lemma 3. *An interval graph G can be $L(h,k)$-labeled with span at most Δh, if $k \le \frac{h}{2}$.*

Proof. Without loss of generality, we focus on connected graphs. We proceed as in Lemma 1 with the difference that the set of labels is $0, h, 2h, \ldots, \Delta h, k, h+k, 2h+k, \ldots, (\Delta-1)h+k$.

Distance 1. Let v_i and v_l be adjacent vertices, assume by contradiction that their labels $l(v_i)$ and $l(v_l)$ differ by less than h. Then only two cases are possible:

(1.1) $l(v_i) = sh$, for some s, and $l(v_l)$ is either $sh+k$ or $(s-1)h+k$. Then $l-i \ge (\Delta-s)+(s-1)+1 = \Delta$. In view of Proposition 3 this is impossible because G has maximum degree Δ. If $i=0$ then l can be at most Δ; hence, $l(v_l)-l(v_i)$ is never smaller than h.

(1.2) $l(v_i) = sh+k$, for some s, and $l(v_l)$ is either sh or $(s+1)h$. Then i cannot be 0 and $l-i \ge (\Delta-1-s)+s+1 = \Delta$. This is in contradiction with Proposition 3.

Distance 2. Let v_i and v_l be vertices at distance two with labels $l(v_i)$ and $l(v_l)$ that differ by less than k. Since $k \le \frac{h}{2}$ it must be $l(v_i) = l(v_l)$, and therefore $l-i = 2\Delta+1$, i.e. the number of the different labels. This contradicts Proposition 3.

From the previous proof it easily follows:

Corollary 1. *If an interval graph G has a canonical order such that the degree of v_0 is strictly less than Δ, G can be $L(h,k)$-labeled with span at most $(\Delta-1)h+k$, if $k \le \frac{h}{2}$.*

The bound stated in the previous lemma is the best possible, as shown by the following:

Theorem 3. *There exists an interval graph requiring at least span Δh to be $L(h,k)$-labeled.*

Proof. Consider $K_{\Delta+1}$, the clique on $\Delta+1$ vertices. As all vertices are adjacent a span of Δh is necessary.

Lemma 4. *An interval graph G can be $L(h,k)$-labeled with span at most $2k\Delta$, if $k \ge \frac{h}{2}$.*

Proof. The proof is analogous to the one of Lemma 3. The only difference is the ordered set of labels used: $0, 2k, 4k, \ldots, 2k\Delta, k, 3k, 5k, \ldots, 2k(\Delta-1)+k$.

Again, it easily follows:

Corollary 2. *If the canonical order of an interval graph G is such that the degree of v_0 is strictly less than Δ, G can be $L(h,k)$-labeled with span at most $2k(\Delta-1)+k$, if $k \ge \frac{h}{2}$.*

Unfortunately, we are not able to exhibit an interval graph requiring at least span $2k\Delta$, if $k \geq \frac{h}{2}$, so it remains an open problem to understand if this result is tight or not.

We can summarize the previous results in the following theorem:

Theorem 4. *An interval graph G can be $L(h,k)$-labeled with span at most* $\max(h, 2k)\Delta$ *and, if G has a canonical order such that the degree of v_0 is strictly less than Δ, G can be $L(h,k)$-labeled with span at most $max(h, 2k)(\Delta - 1) + k$.*

Next theorem allows us to compute another bound for $\lambda_{h,k}(G)$, introducing also the dimension of the maximum clique ω.

Theorem 5. *An interval graph G can be $L(h,k)$-labeled with span at most $min((\omega-1)(2h+2k-2), \Delta(2k-1)+(\omega-1)(2h-2k))$.*

Proof. Consider the greedy algorithm designed as follows:

ALGORITHM `Greedy-Interval`
  consider the canonical order $v_0, v_1, \dots v_{n-1}$
  FOR $i = 0$ TO $n - 1$ DO
    label v_i with the first available label, taking into account
     the labels already assigned to neighbors of v_i
     and to vertices at distance 2 from v_i.

At the i-th step of this $O(n^2)$ time algorithm, consider the vertex set C_i constituted by all the labeled neighbors of v_i and the vertex set D_i constituted by all the labeled vertices at distance 2 from v_i. It is straightforward that $C_i \cap D_i = \emptyset$. As an example consider the graph of Figure 4, when $i = 3$ we have $C_3 = \{v_1, v_2\}$ and $D_3 = \{v_0\}$.

Let v_{min} be the vertex in C_i with the minimum index; in view of the umbrella property for interval graphs, v_{min} is connected to all vertices inside C_i. On the other hand, each vertex v_k in D_i must be adjacent to some vertex in C_i, as it is at distance 2 from v_i; therefore the umbrella property implies that all vertices in D_i are connected to v_{min}. It follows that $\Delta \geq d(v_{min}) \geq |D_i| + (|C_i| - 1) + 1$.

Observe also that both $C_i \cup \{v_i\}$ and $D_i \cup \{v_{min}\}$ are cliques, and hence $|C_i| \leq \omega - 1$ and $|D_i| \leq \omega - 1$.

So, when vertex v_i is going to be labeled, each labeled vertex in C_i forbids at most $2h - 1$ labels and each labeled vertex in D_i forbids at most $2k - 1$ labels. Hence the number f of forbidden labels is at most $|C_i|(2h - 1) + |D_i|(2k - 1)$. About f we can also say:

$f \leq (\omega-1)(2h+2k-2)$ due to the inequalities $|C_i| \leq \omega-1$ and $|D_i| \leq \omega-1$;
$f \leq (|C_i| + |D_i|)(2k - 1) + |C_i|2(h - k) \leq \Delta(2k - 1) + (\omega - 1)(2h - 2k)$ for the inequalities $\Delta \geq |D_i| + |C_i|$ and $|C_i| \leq \omega - 1$.

As the previous reasoning does not depend on i, the maximum span is bounded by $\min((\omega - 1)(2h + 2k - 2), \Delta(2k - 1) + (\omega - 1)(2h - 2k))$.

In Figure 4 it is shown a graph $L(2,1)$-labeled with the greedy algorithm. It is easy to see that in this case the bounds given in the previous theorem, arguments of the min function, coincide and are exactly equal to the required span.

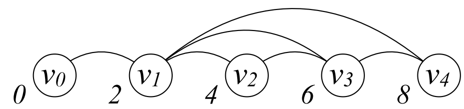

Fig. 4. A graph $L(2,1)$-labeled with the greedy algorithm.

Observe that a trivial lower bound for $\lambda_{h,k}(G)$ is $(\omega-1)h$. So, when $k=1$ the previous theorem provides a 2-approximate algorithm for interval graphs, improving the approximation ratio of [3].

5 Concluding Remarks and Open Problems

In the literature there are no results concerning the $L(h,k)$-labeling of general co-comparability graphs. It is neither known whether the problem remains NP-complete when restricted to this class or to some subclasses, as interval or unit-interval graphs.

In this paper we offered the first known upper bounds for $\lambda_{h,k}$ of co-comparability and interval graphs. The presented proofs are constructive and give the following upper bounds:

$$\lambda_{h,k}(G) \leq \max(h,2k)2\Delta + k$$

if G is a co-comparability graph, and

$$\lambda_{h,k}(G) \leq \max(h,2k)\Delta$$

if G is an interval graph.

Moreover, for interval graphs with certain restrictions, we have reduced this latter bound to $\max(h,2k)(\Delta-1)+k$. We have also shown a greedy algorithm that guarantees, for all interval graphs, a new upper bound on $\lambda_{h,k}(G)$ in terms of both ω and Δ, that is:

$$\lambda_{h,k}(G) \leq \min((\omega-1)(2h+2k-2), \Delta(2k-1)+(\omega-1)(2h-2k))$$

This bound is provided by a 2-approximate algorithm, improving the approximation ratio in [3].

Finally, we have shown that the $L(1,1)$-labeling problem is polynomially solvable for co-comparability graphs.

Many open problems are connected to this research. Here we list just some of them:

– Is the $L(2,1)$-labeling polynomially solvable on co-comparability graphs?
– Is it possible to find some lower bounds to understand how much our results are tight?
– Circular-arc graphs are a natural super-class of interval graphs. Is it possible to extend the results achieved in this paper in order to find an $L(h,k)$-labeling of circular-arc graphs? What about the complexity of the $L(1,1)$-labeling on circular-arc graphs?
– What can we say about the $L(h,k)$-labeling of comparability graphs? It is easy to see that their $L(1,1)$-labeling is polynomially solvable as they are perfect and the square of a comparability graph is still a comparability graph; does the $L(2,1)$-labeling remain polynomially solvable?

Last, but not least, we wish to point out the connection between the linear orderings of co-comparability, interval and unit-interval graphs with a more general concept, namely that of a *dominating pair*, introduced by Corneil, Olariu and Stewart [11]. Considerable attention has been paid to exploiting the linear structure exhibited by various graph families. Examples include interval graphs [23], permutation graphs [15], trapezoid graphs [10, 13], and co-comparability graphs [19].

The linearity of these four classes is usually described in terms of ad-hoc properties of each of these classes of graphs. For example, in the case of interval graphs, the linearity property is traditionally expressed in terms of a linear order on the set of maximal cliques [6, 7]. For permutation graphs the linear behavior is explained in terms of the underlying partial order of dimension two [2], for co-comparability graphs the linear behavior is expressed in terms of the well-known linear structure of comparability graphs [22], and so on.

As it turns out, the classes mentioned above are all subfamilies of a class of graphs called the asteroidal triple-free graphs (AT-free graphs, for short). An independent set of three vertices is called an *asteroidal triple* if between any pair in the triple there exists a path that avoids the neighborhood of the third. AT-free graphs were introduced over three decades ago by Lekkerkerker and Boland [23] who showed that a graph is an interval graph if and only if it is chordal and AT-free. Thus, Lekkerkerker and Boland's result may be viewed as showing that the absence of asteroidal triples imposes the linear structure on chordal graphs that results in interval graphs. Recently, the authors [11] have studied AT-free graphs with the stated goal of identifying the "agent" responsible for the linear behavior observed in the four subfamilies. Specifically, in [11] the authors presented evidence that the property of being asteroidal triple-free is what is enforcing the linear behavior of these classes.

One strong "certificate" of linearity is the existence of a *dominating pair* of vertices, that is, a pair of vertices with the property that every path connecting them is a dominating set. In [11], the authors gave an existential proof of the fact that every connected AT-free graph contains a dominating pair.

In an attempt to generalize the co-comparability ordering while retaining the AT-free property, Corneil, Koehler, Olariu and Stewart [12] introduced the concept of *path orderable graphs.* Specifically, a graph $G = (V, E)$ is path orderable if there is an ordering $v_1, \ldots, v_n$ of its vertices such that for each triple v_i, v_j, v_k with $i < j < k$ and $v_i v_k \notin E$, vertex v_j intercepts each v_i, v_k-path of G; such an ordering is called a *path ordering*.

It is easy to confirm that co-comparability graphs are path orderable. It is also clear that path orderable graphs must be AT-free. It is a very interesting open question whether the results in this paper about the $L(h,k)$-labeling of co-comparability graph can be extended to

- graphs that have an induced dominating pair, and/or
- graphs that are path orderable.

This promises to be an exciting area for further investigation.

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