FREE ITERATIVE THEORIES: A COALGEBRAIC VIEW (EXTENDED ABSTRACT)

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Abstract. Every finitary endofunctor of Set generates a free iterative theory in the sense of Elgot. This is based on coalgebras, specifically on parametric corecursion, and the result holds for more general categories than just Set.

The algebra of (finite and infinite) trees over a given signature can be used to study computations of data through a program, taking into account that such computations are potentially infinite. This topic was extensively studied in the 1970's by Jerzy Tiuryn [T], the ADJ group [ADJ], Calvin C. Elgot [E] and others. Elgot introduced iterative algebraic theories as a technique of studying computation (on Turing machines, say) at a level abstracting from the nature of external memory. The main example presented by Elgot is the theory of regular trees, i.e., infinite trees which posses only finitely many subtrees. He has later proved that this theory is a free iterative theory of a given (finitary) signature Σ.

The purpose of the present paper is to generalize Elgot's result from signatures (=polynomial endofunctors of the category of sets) to finitary endofunctors of Set. Using a very general Solution Theorem, which shows by coalgebraic methods how iterative equations can be solved in categories, developed in previous work, we show that finitary endofunctors generate free iterative theories (=finitary monads).

1. What is a Rational Tree?

The algebra of finite and infinite Σ-labelled trees has, for every signature Σ, the important property that iterative equational systems of a certain (liberal) type have unique solutions. And when we solve flat iterative systems, we obtain precisely the so-called rational trees. Let us give details of this.

For any set X of variables denote by

$$T_\Sigma X$$

the algebra of all finite and infinite Σ-labelled trees with variables from X. That is, trees labelled so that a node with $n > 0$ children is labelled by an n-ary operation symbol (an element of Σ_n) and a leaf is labelled by a variable or a constant (an element of $X + \Sigma_0$). The operations on $T_\Sigma X$ are given by tree-tupling. Now consider a system of iterative equations [1]

$$\begin{array}{rcl} x_0 & \approx & t_0(x_0, x_1, x_2, \dots, y_0, y_1, y_2, \dots) \\ x_1 & \approx & t_1(x_0, x_1, x_2, \dots, y_0, y_1, y_2, \dots) \\ & \vdots & \\ x_n & \approx & t_n(x_0, x_1, x_2, \dots, y_0, y_1, y_2, \dots) \\ & \vdots & \end{array} \tag{1.1}$$

where t_i are trees with variables from $X = \{x_0, x_1, x_2, \dots\}$ and parameters from $Y = \{y_0, y_1, y_2, \dots\}$, i.e.,

$$t_i \in T_\Sigma(X + Y) \quad \text{for } i = 0, 1, 2, \dots$$

Such a system is called *guarded* provided that none of the trees t_i is a variable from X. This condition alone guarantees that there exists a unique *solution* of (1.1), i.e., a unique tuple ${x_i}^\dagger(y_0, y_1, y_2, \dots)$ of trees in $T_\Sigma Y$

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[1] We denote by $\approx$ formal equations and $=$ is the identity of the two sides.

such that the expected identities

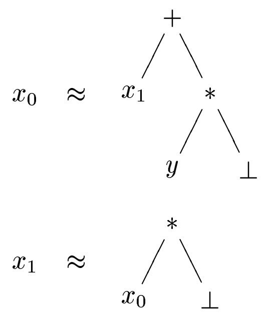
$$\begin{array}{rcl} {x_0}^\dagger & = & t_0({x_0}^\dagger, {x_1}^\dagger, {x_2}^\dagger, \ldots, y_0, y_1, y_2, \ldots) \\ {x_1}^\dagger & = & t_1({x_0}^\dagger, {x_1}^\dagger, {x_2}^\dagger, \ldots, y_0, y_1, y_2, \ldots) \\ & \vdots & \\ {x_n}^\dagger & = & t_n({x_0}^\dagger, {x_1}^\dagger, {x_2}^\dagger, \ldots, y_0, y_1, y_2, \ldots) \\ & \vdots & \end{array}$$

hold.

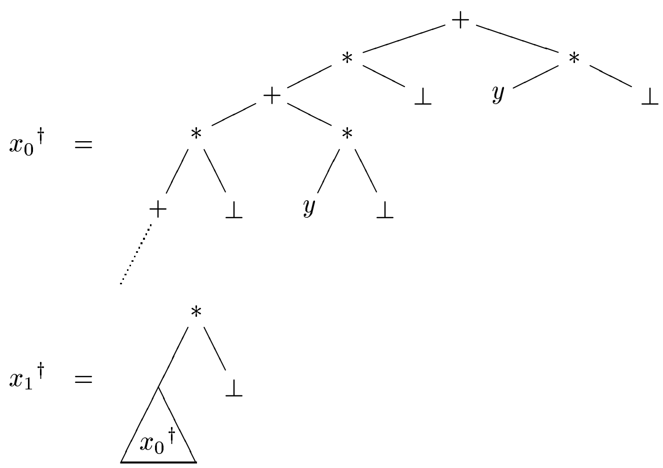
Theorem 1. *Every guarded system of iterative equations has a unique solution.*

In fact, this is a special case of a much more general Solution Theorem we mention in 2 below.

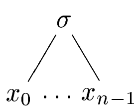
Example 2. Let Σ consist of binary operations $+$ and $*$ and a constant $\perp$. The following system of iterative equations



is guarded. The solution is given by the following trees in $T_\Sigma Y$:



An equational system (1.1) is called *finitary* if it has only finitely many variables, and the right-hand sides are finite trees (such as those in Example 2). Observe that every finitary system has a simple reduction to a *flat* system, i.e., one with finitely many variables, and with right-hand sides which are either

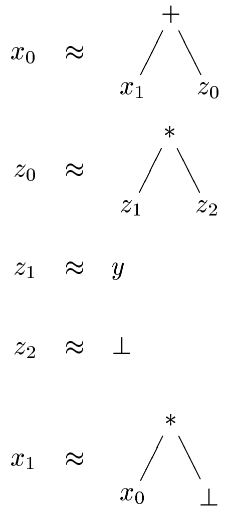
(a) flat trees  with $\sigma \in \Sigma_n$ and $x_0, \ldots, x_{n-1} \in X$

or

(b) single parameters from Y.

Observe that (a) includes single constants $\sigma \in \Sigma_0$.

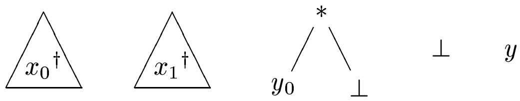
Example 3. A reduction of the system of Example 2 is obtained by introducing new variables, z_0, z_1 and z_2 as follows:



The unique solution of this reduced system is the original solution ${x_0}^\dagger$ and ${x_1}^\dagger$ together with the obvious new trees ${z_i}^\dagger$ for $i = 0, 1, 2$.

Definition 4. A tree in $T_\Sigma Y$ is called *rational* if it can be obtained by solving a flat (or, equivalently, a finitary guarded) system of equations.

Thus, the trees ${x_0}^\dagger$ and ${x_1}^\dagger$ in 2 are examples of rational trees. Rational trees are fully characterized as those trees in $T_\Sigma Y$ that have, up to tree isomorphism, only finitely many subtrees, see [EBT]. For example, all subtrees of ${x_0}^\dagger$ in 2 are isomorphic to one of the following five trees (obtained by breadth-first search of the nodes of ${x_0}^\dagger$):



Now the subalgebra

$$R_\Sigma Y$$

of all rational trees has a solution property "almost as strong" as the algebra $T_\Sigma Y$:

Theorem 5. *Every finite guarded system of iterative equations with rational right-hand sides has a unique solution in $R_\Sigma Y$.*

A direct proof of this theorem is not difficult, but we do not have to discuss it here because we are going to present a much more general result.

2. What is a Solution in General?

We now generalize infinite trees, which are elements of the final coalgebra, $T_\Sigma X$, of the polynomial endofunctor $H_\Sigma(\_) + X$ to final coalgebras of $H(\_) + X$ for an arbitrary endofunctor H. Recall that, given a signature Σ, then the corresponding *polynomial functor*

$$H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$$

defined by

$$H_\Sigma A = \Sigma_0 + \Sigma_1 \times A + \Sigma_2 \times A^2 + \dots$$

has the property that H_Σ-algebras are just the classical universal algebras of signature Σ. A final H_Σ-coalgebra is well-known to be the coalgebra $T_\Sigma \emptyset$ of all finite and infinite Σ-labelled trees without variables. Now the functor $H_\Sigma(\_) + X$ is also polynomial (for the signature obtained from Σ by adding a constant symbol for every variable in X), thus,

$$T_\Sigma X \text{ is a final coalgebra of } H_\Sigma(\_) + X.$$

In [AAMV] the concept of an *iteratable* endofunctor of Set (or, more generally, of any category with binary coproducts) was introduced; it is an endofunctor H such that $H(\_) + X$ has a final coalgebra, for every X. Notation:

$$TX$$

denotes a final coalgebra of $H(\_) + X$, and the coalgebra-structure (which is, by Lambek's Lemma, an isomorphism between TX and $HTX + X$) is denoted by giving names to the coproduct injections of TX (as a coproduct of HTX and X) as follows:

$$\eta_X : X \longrightarrow TX \quad (\text{"injection of variables"})$$

and

$$\tau_X : HTX \longrightarrow TX \quad (\text{"}TX\text{ becomes an }H\text{-algebra"}).$$

Definition 6. By a *guarded equation morphism* for H is meant a morphism of the following form

$$e : X \longrightarrow T(X+Y)$$

(X is the object of variables and Y the object of parameters) which factors through $[\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}] : HT(X+Y) + Y \longrightarrow T(X+Y)$:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, dotted] & T(X+Y) \\ & HT(X+Y)+Y \arrow[u, "{[\tau_{X+Y}, \eta_{X+Y}\cdot \mathsf{inr}]}"'] \end{tikzcd}$$

This is precisely a guarded system (1.1) above in case H is the polynomial endofunctor H_Σ. In fact, the equations $x_i \approx t_i$, where $t_i \in T(X+Y)$, just form a function $e : X \longrightarrow T(X+Y)$, $x_i \mapsto t_i$. And since $T(X+Y) = \big(HT(X+Y)+Y\big)+X$ is a coproduct with injections $[\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}]$ and $\eta_{X+Y} \cdot \mathsf{inl} : X \longrightarrow T(X+Y)$, to say that the right-hand sides t_i are never variables from X is precisely to say that e factors through $[\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}]$.

The H-algebras TX have a rich structure, as discovered independently by Larry Moss [M], and the present author together with Peter Aczel, Jiří Adámek and Jiří Velebil [AAMV]. Firstly, the way how substitution of trees in $T_\Sigma X$ for variables is well known to work generalizes to all iteratable endofunctors:

7 Substitution Theorem. *For every morphism $s : X \longrightarrow TY$ there exists a unique homomorphism $\widehat{s} : TX \longrightarrow TY$ of H-algebras extending s (i.e., with $s = \widehat{s}\eta_X$).*

Next, we can introduce solutions for equation morphisms $e : X \longrightarrow T(X+Y)$ by mimicking the case of trees as follows: a solution of e is a morphism $e^\dagger : X \longrightarrow TY$. It has the property that when the following "substitution" morphism

$$s = [e^\dagger, \eta_Y] : X + Y \longrightarrow TY$$

is considered (i.e., every variable x_i is substituted by $e^\dagger(x_i) = {x_i}^\dagger \in TY$, while parameters are unchanged), then the composite

$$X \xrightarrow{\ e\ } T(X+Y) \xrightarrow{\ \widehat{s}\ } TY$$

(corresponding to performing the substitution s on all variables on the right-hand sides of (1.1)) is equal to $e^\dagger$.

Definition 8. By a *solution* of an equation morphism $e : X \longrightarrow T(X+Y)$ is meant a morphism $e^\dagger : X \longrightarrow TY$ such that the following triangle

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & TY \\ T(X+Y) \arrow[ur, "\widehat{[e^\dagger, \eta_Y]}"'] & \end{tikzcd}$$

commutes.

The following result is called Parametric Corecursion in [M]:

9 Solution Theorem. *Given an iteratable endofunctor H, every guarded equation morphism has a unique solution.*

It follows trivially from the Substitution Theorem that $(T, \eta, \widehat{(\_)})$ is a *Kleisli triple*, i.e., that the following three conditions are satisfied:

(i) $\widehat{\eta_X} = id_{TX}$, for every object X,
(ii) $\widehat{s}\eta_X = s$, for every morphism $s : X \longrightarrow TY$,

and

(iii) $\widehat{r}\widehat{s} = \widehat{\widehat{r}s}$, for every pair $s : X \longrightarrow TY$ and $r : Y \longrightarrow TZ$.

In other words, we obtain a monad (T, η, μ), where $\mu_X = \widehat{id_{TX}} : TTX \longrightarrow TX$. However, this monad is usually not finitary; thus, it cannot be identified with an algebraic theory in the sense of Lawvere.

Example 10. For every signature Σ we have a monad (T_Σ, η, μ) where $T_\Sigma X$ is the algebra above, η is the insertion of variables (as one-node trees), and μ is the usual substitution of trees into trees. This monad is seldom finitary. (In fact, it is finitary iff all operations of Σ are either unary or nullary.)

3. What is a Rational Monad?

Consider a *finitary* endofunctor H of Set, i.e., one that preserves filtered colimits. Then H is iteratable (see [AAMV]).

Definition 11. By a *flat equation morphism* for H is meant a morphism of the following form

$$e : X \longrightarrow HX + Y \quad (X \text{ finite}).$$

Shortly, flat equation morphisms are just the finite coalgebras of $H(\_) + Y$. (If $H = H_\Sigma$, then a flat equation morphism

$$e : X \longrightarrow \coprod_{n \in \omega} \Sigma_n \times X^n + Y \quad (X \text{ finite})$$

is precisely a flat equation system as introduced above.) Every flat equation morphism gives rise to a guarded equation morphism as follows:

$$X \xrightarrow{\quad e \quad} HX + Y \xrightarrow{H\eta_X + Y} HTX + Y \xrightarrow{[T\mathsf{inl}\cdot\tau_X, T\mathsf{inr}\cdot\eta_Y]} T(X + Y)$$

By an innocent abuse of notation, we denote by $e^\dagger : X \longrightarrow TY$ the unique solution of the latter. For flat equation morphisms, we have

$$\text{solution} = \text{corecursion}.$$

More precisely, we obtain the following result.

Proposition 12. *For any flat equation morphism e, the arrow $e^\dagger$ is the unique homomorphism of the coalgebra X into the final coalgebra TY.*

We mimick the definition of $R_\Sigma Y$ as the algebra of all solutions of flat equation systems (for $H = H_\Sigma$) in our categorical setting. For any set Y denote by $\mathcal{EQ}_Y$ the category of flat equation morphisms, i. e., the full subcategory of the category of coalgebras of $H(\_) + Y$ given by taking the finite coalgebras as objects. Observe that this is an essentially small, filtered category. Indeed, $\mathcal{EQ}_Y$ is finitely cocomplete (since Set is) and has up to isomorphism only a set of objects. Furthermore, denote by

$$\mathrm{EQ}_Y : \mathcal{EQ}_Y \longrightarrow \mathsf{Set}$$

the restriction of the forgetful functor and let

$$RY = \operatorname{colim} \mathrm{EQ}_Y.$$

It turns out that RY is an H-algebra. In fact, RY is a coproduct of HRY and Y, that means that there are coproduct injections

$$\begin{aligned} \eta^R_Y &: Y \longrightarrow RY && (\text{“injection of variables”}) \\ \varrho_Y &: HRY \longrightarrow RY && (\text{“}RY \text{ is an } H\text{-algebra”}). \end{aligned}$$

The H-algebras RY, which are subalgebras of TY, also have a rich structure; we obtain the following

13 Rational Substitution Theorem. *For every morphism $s : X \longrightarrow RY$ there exists a unique homomorphism $\widetilde{s} : RX \longrightarrow RY$ of H-algebras extending s, (i. e. with $s = \widetilde{s}\eta^R_X$).*

Analogously as for T one can now define the notions of guarded equation morphism and solution for R. A *rational* equation arrow is an arrow

$$X \longrightarrow R(X + Y).$$

Such an equation arrow is guarded if it factors as shown in the following diagram

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, dotted] & R(X+Y) \\ & HR(X+Y) + Y \arrow[u, "{[\varrho_{X+Y}, \eta^R_{X+Y}\cdot\mathsf{inr}]}"'] \end{tikzcd}$$

A solution for e is an arrow $e^\dagger : X \longrightarrow RY$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & RY \\ R(X+Y) \arrow[ur, "\widetilde{[e^\dagger, \eta^R_Y]}"'] & \end{tikzcd}$$

commutes. We obtain the

14 Rational Solution Theorem. *Every rational guarded equation morphism $e : X \longrightarrow R(X+Y)$ has a unique solution $e^\ddagger : X \longrightarrow RY$.*

As a corollary of the above Substitution Theorem we conclude immediately that R is also a monad on Set, this time a finitary monad (or, equivalently, Lawvere's algebraic theory). We call this monad the *rational monad* generated by the given finitary endofunctor H.

4. Rational Monad is a Free Iterative Monad

C. Elgot has studied algebraic theories (=finitary monads on Set) with the property that certain iterative equations have unique solutions; he called such theories iterative. Based on 1, it is quite natural to call, for a given monad (S, η, μ), any morphism

$$e : X \longrightarrow S(X+Y)$$

an equation morphism (for S). Recall that every monad defines substitution: for every $s : X \longrightarrow SY$ we have the corresponding homomorphism $\widehat{s} = \mu_Y \cdot Ss : SX \longrightarrow SY$ of the free S-algebras. Now, a morphism $e^\dagger : X \longrightarrow SY$ will be called a *solution* of e provided that the following triangle

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & SY \\ S(X+Y) \arrow[ur, "\widehat{[e^\dagger, \eta_Y]}"'] & \end{tikzcd}$$

commutes. But how do we want to express the property of e being guarded? (Or ideal, which is a slightly stricter concept that C. Elgot used: for trees this means that the right-hand sides are neither variables, nor single parameters. But this difference is inessential.) For that purpose, C. Elgot has introduced the concept of an ideal algebraic theory. Translated into the language of monads, it yields the following (see [AAMV] for a very simple proof that our formulation is equivalent to Elgot's):

Definition 15. A monad (S, η, μ) is called *ideal* provided that

(i) $S = S' + Id$ and $\eta : Id \longrightarrow S$ is the right-hand injection (notation: $\sigma : S' \longrightarrow S$ for the left-hand one)

and

(ii) μ restricts to $\mu' : S'S \longrightarrow S'$, i.e., there is a natural transformation μ' such that the following square

$$\begin{tikzcd} S'S \arrow[r, "\mu'"] \arrow[d, "\sigma S"'] & S' \arrow[d, "\sigma"] \\ SS \arrow[r, "\mu"'] & S \end{tikzcd}$$

commutes.

Furthermore, for any iterative monads $S = S' + Id$ and $U = U' + Id$ we call a monad morphism $\alpha : S \longrightarrow U$ *ideal* if $\alpha = \alpha' + Id$ for some natural transformation $\alpha' : S' \longrightarrow U'$.

Example 16. The monad T defined by 9 is ideal: here $T = HT + Id$ with injections τ (=σ) and η. Also, the monad R defined by Rational Solution Theorem is ideal.

For ideal monads (S, η, μ) we call an equation morphism $e : X \longrightarrow S(X+Y)$ *guarded* provided that it factors through $[\sigma_{X+Y}, \eta_{X+Y}\mathsf{inr}] : S'(X+Y)+Y \longrightarrow S(X+Y)$:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, dotted] & S(X+Y) \\ & S'(X+Y)+Y \arrow[u, "{[\sigma_{X+Y}, \eta_{X+Y}\cdot\mathsf{inr}]}"'] \end{tikzcd}$$

Definition 17. A finitary monad S is called *iterative* provided that

(i) S is ideal

and

(ii) every guarded equation morphism $e : X \longrightarrow S(X+Y)$ with X finite has a unique solution.

The main result of this paper is the fact that every finitary endofunctor of Set generates a free iterative monad, viz, the rational monad.

Theorem 18. *The rational monad R is the free iterative monad over H, i. e., for any iterative monad $S = S' + Id$ and any natural transformation $\lambda' : H \longrightarrow S'$ there exists a unique ideal monad homomorphism $\overline{\lambda} = \overline{\lambda}' + Id : R \longrightarrow S$ such that $\overline{\lambda}' H\eta^R = \lambda'$:*

$$\begin{tikzcd} H \arrow[r, "H\eta^R"] \arrow[dr, "\forall\lambda'"'] & HR \arrow[d, "\overline{\lambda}'"] & R \arrow[d, dashed, "\exists!\overline{\lambda}"] \\ & S' & S. \end{tikzcd}$$

Note that this last result is analogous to the main result in [AAMV] concerning *completely iterative monads*, as introduced in [EBT]: they are defined as above just dropping the requirement that X be finite. In [AAMV] it is proved that every iteratable endofunctor generates a free completely iterative monad, viz, the above monad T.

5. Beyond the Category of Sets

There are two principle features in which the present results differs from those in [AAMV]: the proofs are more technically involved and there are numerous side conditions on $\mathcal{A}$ and H that need to be assumed: $\mathcal{A}$ has to be locally finitely presentable and extensive (i. e. coproducts are disjoint and universal; cf. [CLW]), the classes of finitely presentable und finitely generated objects need to coincide, and $\mathcal{A}$ needs to have finite hom-sets for finitely presentable objects. Finally, H has to preserve monomorphisms. The side conditions on $\mathcal{A}$ are quite restrictive but still allow to apply our results to other categories than Set. One example is Pos, the category of posets and monotone maps.

In contrast, our description of a free completely iterative monad in [AAMV] was dramatically simpler than the proof for H_Σ presented in [EBT] (and involving [E]), and yet, the only assumption was that $\mathcal{A}$ have binary coproducts with monomorphic coproduct injections.

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References

[AAM] J. Adámek, J. Velebil and S. Milius, Free Iterative Theories: A Coalgebraic View, *manuscript*

[AAV] P. Aczel, J. Adámek and J. Velebil, A Coalgebraic View of Infinite Trees and Iteration, *Electronic Notes in Theoretical Computer Science*, 44.1 (2001)

[AAMV] P. Aczel, J. Adámek, S. Milius and J. Velebil, Infinite Trees and Completely Iterative Theories: A Coalgebraic View, *manuscript*

[A97] J. Adámek, A Categorical Generalization of Scott Domains, *Math. Struct. in Comp. Science* 7 (1997), 419–443

[ADJ] J. A. Goguen, S. W. Thatcher, E. G. Wagner and J. B. Wright, Initial Algebra Semantics and Continuous Algebras, *Journal of the ACM* 24 (1977), 68–95

[AMV] J. Adámek, S. Milius and J. Velebil, Finitary Kleisli Triples, *manuscript*

[ARo] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, Cambridge University Press, 1994

[AT] J. Adámek and V. Trnková, *Automata and Algebras in Categories*, Kluwer Academic Publishers, Dordrecht, 1990

[CLW] A. Carboni, S. Lack and R. F. C. Walters, Introduction to Extensive and Distributive Categories, *J. Pure Appl. Algebra*, **84**, (1993), 145–158

[E] C. C. Elgot, Monadic Computation and Iterative Algebraic Theories, in: *Logic Colloquium '73* (eds: H. E. Rose and J. C. Shepherdson), North-Holland Publishers, Amsterdam, 1975

[EBT] C. C. Elgot, S. L. Bloom, R. Tindell, On the Algebraic Structure of Rooted Trees, *J. Comp. Syst. Sciences*, **16**, (1978), 361–399

[M] L. Moss, Parametric Corecursion, *preprint*, available at `http://math.indiana.edu/home/moss/parametric.ps`

[T] J. Tiuryn, Unique Fixed Points vs. Least Fixed Points, *Theoretical Computer Science 12* (1980), 229–254

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