DIGITAL CONVEXITY AND FIXED POINT PROPERTY

Rueiher Tsaur, Michael B. Smyth[1]

Abstract. We extend some of our previous results on fixed points of graph multifunctions to digraphs and, especially, posets. The posets of most interest here are the (finite) Khalimsky spaces, in their specialization order. Notions of convexity can be defined in these spaces, providing the basis for certain "geometric" fixed point theorems (related to the classical Brouwer and Kakutani theorems).

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1. Preliminaries and notations

By a *graph* G we mean a set $V(G)$ of *vertices*, with a reflexive and symmetric relation $E(G)$ the *edges*, also known as a *tolerance* graph in [8]. For graphs G and H, a *graph homomorphism* $f : G \to H$ maps vertices of G to vertices of H, preserving the relation, i.e., $(f(x), f(y)) \in E(H)$ whenever $(x, y) \in E(G)$. By a *multifunction* $f : G \to H$ we understand a mapping that assigns to a vertex x of G a non-empty subset $f(x)$ in H.

Definition 1.1. Let G be a graph. The *weak* and *strong power* graphs of G, $\mathcal{P}_w(G)$ and $\mathcal{P}_s(G)$, have the non-empty finite subsets of $V(G)$, $\mathcal{P}^+_{fin}(V(G))$ as vertices, such that for any $A, B \in \mathcal{P}^+_{fin}(V(G))$

$$(A,B) \in E(\mathcal{P}_w(G)) \Leftrightarrow \exists x \in A, y \in B, (x,y) \in E$$

$$(A,B) \in E(\mathcal{P}_s(G)) \Leftrightarrow \forall x \in A, \exists y \in B, (x,y) \in E \;\&\; \forall y \in B, \exists x \in A, (x,y) \in E.$$

Definition 1.2. Let G and H be graphs. The multifunction f from G to H, $f : G \to H$, is *weak* (resp. *strong*) if $\hat{f} : G \to \mathcal{P}_w(H)$ (resp. $\hat{f} : G \to \mathcal{P}_s(H)$) is a graph homomorphism, where $\hat{f}(x) = f(x) \in \mathcal{P}^+_{fin}(V(H))$.

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[1] Department of Computing, Imperial College, London SW7 2BZ; e-mail: `rt2@doc.ic.ac.uk` & `mbs@doc.ic.ac.uk`

In [13,14] the power constructions for simplicial complexes were also developed, and, in terms of them, the above two concepts of multifunctions were extended to simplicial multifunctions. As explained in the previous works, multifunctions are appropriate for approximating classical continuous functions.

Definition 1.3. Let G be a graph and $f : G \to G$ a self-mapping multifunction. Then a vertex $x \in V(G)$ is said to be an *almost fixed point* of f if $(\{x\}, f(x)) \in E(\mathcal{P}_w(G))$ (that is, x is adjacent to some element of $f(x)$).

Let ϕ be a property of sets of vertices of the graph G, e.g., graph convexity of G. Then G is said to have the *ϕ-almost fixed point property* (ϕ-AFPP) if for any self-mapping multifunction $f : G \to G$ which sends each vertex of $V(G)$ into a subset of G satisfying ϕ (f is said to be a *ϕ-multifunction* in this case), f has an almost fixed point.

Given a graph G. For any subset $A \subseteq V(G)$, we denote by $G[A]$ and $G\backslash A$ ($G\backslash x$ if x is a singleton), the subgraphs of G induced by A and $V(G) - A$ respectively. If G is connected, for any two vertices $x, y \in V(G)$, a *path* joining x, y is a (sub)graph $l = \langle v_0, \dots, v_k \rangle$, $V(l) = \{v_0, \dots, v_k\}$, $v_i \neq v_j$ if $i \neq j$, and $E(l) = \{(v_i, v_{i+1}) \mid 0 \leq i < k\}$, such that $v_0 = x$ and $v_k = y$. Let d_G be the *canonical* (graph) distance of G. The *k-ball*, for any $k \in \mathbb{N}$, centered at a vertex $x \in V(G)$ of (the connected graph) G, is the set $B_G(x, k) = \{y \in V(G) \mid d_G(x, y) \leq k\}$.

A vertex x in a graph G is called a *subdominant vertex* of G if there is a vertex y in G, distinct from x, such that $B_G(x, 1) \subseteq B_G(y, 1)$. In this case we will say that y *dominates* x, or that x is *dominated* by y. A finite connected graph G is said to be *dismantlable* if the vertices of G can be listed as $x_1, x_2, \dots, x_m$, where $m = \#P$, such that, for all $i = 1, 2, \dots, m-1$, x_i is a subdominant vertex of the induced subgraph $G[V\backslash\{x_j \mid j < i\}]$.

2. Dismantlable posets and graphs

Given a partially ordered set (poset for short in the sequel) $(P, \leq)$. For elements x, y of P, we say that y *covers* x in P if $x < y$ and there is no element z of P for which $x < z < y$. In this case, x is called a *lower cover* of y in P, and y is called an *upper cover* of x in P. As in [12], an element a of P is said to be an *irreducible* element of P if either a has exactly one upper cover in P or a has exactly one lower cover in P. P is said to be *dismantlable* if the elements of P can be listed as $x_1, x_2, \dots, x_m$, where $m = \#P$, such that, for all $i = 1, 2, \dots, m-1$, x_i is an irreducible element of the poset $P\backslash\{x_j \mid j < i\}$.

Let P and Q be posets. A function $f : P \to Q$ is said to be *monotone* (*order-preserving*) if, for all x, y in P, $x \leq y$ implies $f(x) \leq f(y)$. A multifunction $g : P \to Q$ is said to be *strong* (*isotone*) if, for all x, y in P, $x \leq y$ implies that $\forall u \in g(x), \exists v \in g(y), u \leq v$ and $\forall v \in g(y), \exists u \in g(x), u \leq v$. (In other words, g maps P to $\mathcal{P}(Q)$ taken with the Egli-Milner ordering.) The poset P is said to have the *fixed point property* (FPP) if every monotone function $f : P \to P$ has a fixed point (that is, there is an element $x \in P$ such that $x = f(x)$). Let ϕ be a property of sets of elements of the poset P. Then P is said to have the *ϕ-fixed point* *property* (ϕ-FPP) if for any self-mapping multifunction $f : P \to P$ which sends each element of P into a subset of P satisfying ϕ (f is said to be a *ϕ-multifunction* in this case), f has a fixed point (that is, there is an element $x \in P$ such that $x \in f(x)$).

Theorem 2.1 ([15, 16]). *Every dismantlable graph has the almost fixed point property for strong multifunctions (strong AFPP).*

In [17], Walker showed that every finite dismantlable poset has the strong (isotone or relational) FPP.

Theorem 2.2. *Theorem 2.1 implies Walker's result on the strong FPP for finite dismantlable posets.*

For any poset $(P, \leq)$, we let $\mathrm{Comp}(P)$ denote the *comparability graph* of P. This is the graph whose vertex set is P, and in which two vertices x, y of $\mathrm{Comp}(P)$ are adjacent if x and y are comparable in P (that is, if either $x < y$ or $y < x$). For any graph G, we let $\mathrm{Clique}(G)$ denote the *clique poset* of G. This is the poset whose elements are all non-empty cliques in G such that two elements x, y of $\mathrm{Clique}(G)$ satisfy $x \leq y$ if and only if $x \subseteq y$ in G.

In [4], it was shown that for any poset P, P is dismantlable if and only if $\mathrm{Comp}(P)$ is a dismantlable graph.

Proof. Let P be any finite dismantlable poset, and $f : P \to P$ a strong multifunction. Define a multifunction $g : \mathrm{Comp}(P) \to \mathrm{Comp}(P)$ by $g(x) = f(x), \forall x \in P$. It is clear that g is a strong multifunction. By Theorem 2.1, g has an almost fixed point, say $x \in P$. Hence there exists an element $y \in f(x)$ such that $x \leq y$ or $y \leq x$. Then by Höft's Theorem 2.2 (and its dual theorem) in [5], f has a fixed point. □

3. An almost fixed point theorem for digraphs

By a *digraph* D we mean a set $V(D)$ of vertices, with a reflexive relation $E(D)$ the *arrows*. It is clear that digraphs generalize the (tolerance) graphs and posets. For digraphs D_1 and D_2, a *digraph homomorphism* $f : D_1 \to D_2$ maps vertices of D_1 to vertices of D_2, preserving the (directed) relation, i.e., $\overrightarrow{f(x)f(y)} \in E(D_2)$ whenever $\overrightarrow{xy} \in E(D_1)$.

The *outset* of a vertex x of a digraph D is the set $B_D^+(x) = \{y \mid \overrightarrow{xy} \in E(D)\}$. Similarly, $B_D^-(x) = \{y \mid \overrightarrow{yx} \in E(D)\}$ is the *inset* of x.

Definition 3.1. Let D be a digraph. Then D is said to be *dismantlable* if there is a linear ordering $<$ on $V(D)$, say $v_1 < v_2 < \ldots < v_k$, such that, for all $i \leq k-1$, there exists a v differing from v_i, in the induced sub-digraph $D_i = D[\{x_j \in X \mid j \geq i\}]$, satisfying $B_D^+(v_i)\backslash v_i \subseteq B_D^+(v)\backslash v$, $B_D^-(v_i)\backslash v_i \subseteq B_D^-(v)\backslash v$ and $\overrightarrow{v_iv} \in E(D)$ or $\overrightarrow{vv_i} \in E(D)$ (in this case, we say that v *dominates* v_i, or that v_i is *dominated* by v).

Clearly, Definition 3.1 generalizes the notion of dismantlability of (tolerance) graphs and posets.

Theorem 3.2. *Every dismantlable digraph D has the almost fixed point property (AFPP) for strong multifunctions (w.r.t. arrow directions).*

Proof. Let D be a dismantlable digraph and $f : D \to D$ a self-mapping strong multifunction. Since D is dismantlable, there exist $x, y \in V(D)$ such that $B^+_D(x)\backslash x \subseteq B^+_D(y)\backslash y$, $B^-_D(x)\backslash x \subseteq B^-_D(y)\backslash y$ and $\overrightarrow{xy} \in E(D)$ or $\overrightarrow{yx} \in E(D)$ (**Condition 1**), and the induced subgraph $D\backslash x$ is a dismantlable digraph (**Condition 2**).

We prove by induction on $\#V(D)$ that f has an almost fixed point. *Initial step:* It is clear that D has the AFPP when $\#V(D) = 1$ or 2. *Inductive step:* We assume that D has the AFPP when $\#V(D) \le m \in \mathbb{N}$, $m \ge 1$, and we claim that D has the AFPP when $\#V(D) = m+1$. Define the multifunction $g : D\backslash x \to D\backslash x$ by

$$g(u) = \begin{cases} f(u), & x \notin f(u) \\ (f(u)\backslash x) \cup \{y\}, & x \in f(u). \end{cases}$$

We claim that g is a strong multifunction. Indeed, let $a, b \in V(D\backslash x)$ such that $\overrightarrow{ab} \in E(D\backslash x)$. We have $\overrightarrow{ab} \in E(D)$ and hence $f(a)$ and $f(b)$ are strongly adjacent in D. Now $g(a), g(b)$ are obtained from $f(a), f(b)$ by replacing any occurrences of x by y; since $B^+_D(x)\backslash x \subseteq B^+_D(y)\backslash y$ and $B^-_D(x)\backslash x \subseteq B^-_D(y)\backslash y$, it is clear that such a replacement cannot cause strong relation of $D\backslash x$ to fail. Thus g is a strong multifunction in $D\backslash x$.

Therefore by our inductive assumption and **Condition 2**, there exists a $z \in V(D\backslash x)$ such that z is an almost fixed point of g. Then (since $f(z)$ differs from $g(z)$ by, at most, having x in place of y) z must be an almost fixed point also of f, except in the following case: y is the only vertex of $g(z)$ adjacent to $z, x \in f(z)$, but $y \notin f(z)$, see Fig. 1. Since $\overrightarrow{yz} \in E(D)$ or $\overrightarrow{zy} \in E(D)$, and $f(y)$ and $f(z)$ are strongly adjacent, we consider the following cases:

(1) $\overrightarrow{yz} \in E(D)$: so $f(y)$ contains a vertex, say w, such that $\overrightarrow{wx} \in E(D)$, hence $\overrightarrow{wy} \in E(D)$. Thus y is an almost fixed point of f.

(2) $\overrightarrow{zy} \in E(D)$: so $f(y)$ contains a vertex, say w, such that $\overrightarrow{xw} \in E(D)$, hence $\overrightarrow{yw} \in E(D)$. Thus y is an almost fixed point of f. □

4. Helly property

Recall that a family F of subsets of a set X is said to satisfy the (2-)*intersection property* if any two elements of F are intersection non-empty (also called pairwise non-disjoint). F satisfies the *Helly property* if for every subfamily F' of F satisfying the intersection property, F' is intersection non-empty. We say that a finite connected graph G is a *Helly graph* if the family of balls $B_G(x, k), \forall x \in V(G), \forall k \in \mathbb{N}$, satisfies the Helly property [10].

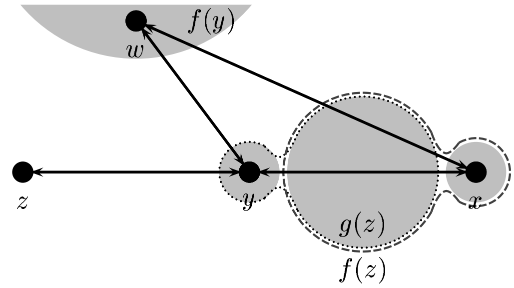

FIGURE 1. $x, y, z, w, g(z), f(z)$ and $f(y)$

A *graph convexity* [2,3] on a connected graph G is an ordered pair $(V(G), \mathcal{C})$, where $\mathcal{C} \subseteq \mathcal{P}(V(G))$ is an algebraic closure system of subsets of $V(G)$ called *convex sets*, such that

(c1) $\emptyset, V(G) \in \mathcal{C}$
(c2) $\bigcap D \in \mathcal{C}$ for any $D \subseteq \mathcal{C}$
(c3) $\bigcup T \in \mathcal{C}$ for any chain $T \subseteq \mathcal{C}$
(c4) C induces a connected subgraph of G for any $C \in \mathcal{C}$.

Let G be a (finite) connected graph, and M the family of balls of G, $M = \{B_G(x,k) \mid x \in V(G), k \in \mathbb{N}\}$. Then the family of subsets of $V(G)$, denoted by $\blacksquare(G)$, which contains M and all finite intersections of elements of M, is a graph convexity if G is Helly [16]. In the following we call $\blacksquare(G)$ the *neighbourhood convexity* of the graph G.

It is shown that every Helly graph has the neighbourhood convexity p-AFPP for p-weak multifunctions [16]. In the following we extend this result to posets (more precisely, Khalimsky spaces).

The so-called Khalimsky (integer) line is defined to be the set of all integers $\mathbb{Z}$ with its natural order together with its interval alternating topology T_{ia}, where T_{ia} is a topology defined by sets of types $(-\infty, 2i], [2j, \infty), i, j \in \mathbb{Z}$ (or alternately $(-\infty, 2i-1], [2j-1, \infty), i, j \in \mathbb{Z}$) as a subbasis. The Khalimsky (digital) arc then can be defined as a finite connected subspace of the Khalimsky line. The n-dimensional Khalimsky space is a topological product of n arcs [6,7].

Using the specialization order of a Khalimsky space, we can view Khalimsky space as a poset (or directed graph). For example, Fig. 2 shows a 2-dimensional Khalimsky space.

The strong product (n times) of m-paths $I_m = \langle 0, \ldots, m \rangle$ is denoted by $I_{m,n}$. The neighbourhood convex sets are, in effect, the rectangular blocks in $I_{m,n}$. A maximal clique of $I_{m,n}$ has 2^n vertices, and is called an *elementary n-cube* of $I_{m,n}$. Let $K_{m,n}$ be the poset whose elements are the neighbourhood convex sets which are contained in elementary n-cubes, ordered by the opposite of inclusion. Then we have

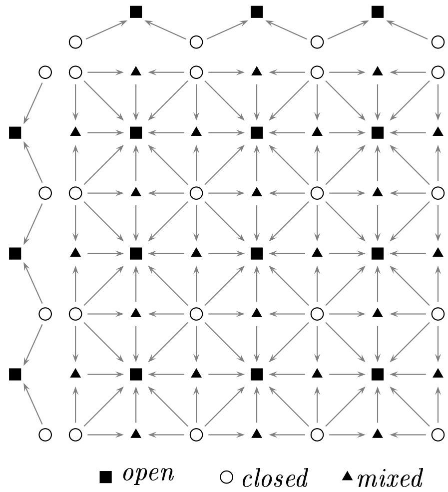

FIGURE 2. A two dimensional Khalimsky space

Theorem 4.1. *$K_{m,n}$ is an n-dimensional Khalimsky space, that is, the n-product of Khalimsky arcs each of which has m open points and $m+1$ closed points (for $m \geq 1$).*

Conversely, given an n-dimensional Khalimsky space $K = K_{m,n}$, let the graph $J_{m,n}$ be its "strong compatibility graph". More precisely, the vertices of $J_{m,n}$ are the points of $K_{m,n}$, and x, y are adjacent in $J_{m,n}$ if $\uparrow x \cap \uparrow y \neq \emptyset$ and $\downarrow x \cap \downarrow y \neq \emptyset$ in $K_{m,n}$. Then we have:

Lemma 4.2. $J_{m,n} \cong I_{2m,n}$.

A *code* is a finite sequence $(a_1, \ldots, a_k)$ of numbers of the set $\{-1, 1\}$ [9]. Let $P = \langle x = x_0, x_1, \ldots, x_k = y \rangle$ be a path between two vertices x and y in a digraph D such that x_i and x_{i+1} are different, for $i < k$. We say that the code $(a_1, \ldots, a_k)$ is associated with P if we have

$$a_i = \begin{cases} 1, & \overrightarrow{x_{i-1}x_i} \in E(D) \\ -1, & \overrightarrow{x_i x_{i-1}} \in E(D). \end{cases}$$

We denote by $C_D(x, y)$ the set of all codes σ such that there exists a path P in D between x and y, with σ associate to P.

For a code σ, we denote by σ^+ and σ^- respectively the numbers of $+1$ and -1 in σ. Let K be a Khalimsky space. For any elements x and y of K, we define two distance functions $d_K^+, d_K^- : K \times K \to \mathbb{N}$ by $d_K^+(x, y) = \min\{\sigma^+ \mid \sigma \in C_K(x, y)\}$ and $d_K^-(x, y) = \min\{\sigma^- \mid \sigma \in C_K(x, y)\}$. Note that d_K^+ and d_K^- are quasi metrics.

For any vertex x of K, we say that $B_K^+(x,k) = \{y \mid d^+(x,y) \leq k\}$ is a *k-closed* ball of x. Similarly, $B_K^-(x,k) = \{y \mid d^-(x,y) \leq k\}$ is a *k-open* ball of x. Let $M^+(K)$ (resp. $M^-(K)$) be the family of closed (resp. open) balls of K. Then the family of subsets of K, denoted by $\blacksquare^+(K)$ (resp. $\blacksquare^-(K)$), which contains $M^+(K)$ (resp. $M^-(K)$) and all finite intersections of elements of $M^+(K)$ (resp. $M^-(K)$), have the following property

Lemma 4.3. $\blacksquare^+(K)$ *and* $\blacksquare^-(K)$ *are graph and order convexities.*

In fact, it is easy to see that $\blacksquare^+(K)$ (resp. $\blacksquare^-(K)$) is a collection of closed (resp. open) neighbourhood (in the sense of topology, that is, for any $C \in \blacksquare^+(K)$ and $D \in \blacksquare^+(K)$, C is a down set and D is an up set) of K. We may call $\blacksquare^+(K)$ and $\blacksquare^-(K)$ respectively the closed and open neighbourhood convexity of K.

Theorem 4.4. *K has the closed neighbourhood convexity FPP for weak multifunctions.*

Proof. Let $f : K \to K$ be a self-mapping weak multifunction mapping each element of K to an element of $\blacksquare^+(K)$. Assume $K = K_{m,n}$, then $J_{m,n} \cong I_{2m,n}$. Define a multifunction $g : J_{m,n} \to J_{m,n}$ by $g(x) = f(x)$. It is clear that g is weak since f is weak. Furthermore, $g(x)$ is a neighbourhood convex set of $J_{m,n}$ for any vertex $x \in V(J_{m,n})$. Hence g has a fixed clique, say $\Delta \subseteq V(J_{m,n})$ (see [16]). Consider the corresponding subset $\Delta' \subseteq K_{m,n}$ of Δ. If Δ' is a chain, then clearly f has a fixed point. If Δ' is incomparable (for some elements in Δ'), then it is not difficult to show that the minimal element $w \in \downarrow \Delta'$ is a fixed point of f. $\square$

Corollary 4.5. *K has the open neighbourhood convexity FPP for weak multifunctions.*

The famous Brouwer's fixed point theorem states that every closed n-topological ball has the fixed point property. By an argument similar to that given in [16]:

Theorem 4.6. *Theorem 4.4 and Corollary 4.5 imply Brouwer's fixed point theorem.*

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