Displaying blocking pairs in signed graphs

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Abstract

A signed graph is a pair (G, Σ) where G is a graph and Σ is a subset of the edges of G. A circuit of G is even (resp. odd) if it contains an even (resp. odd) number of edges of Σ. A *blocking pair* of (G, Σ) is a pair of vertices s, t such that every odd circuit intersects at least one of s or t. In this paper, we characterize when the blocking pairs of a signed graph can be represented by 2-cuts in an auxiliary graph. We discuss the relevance of this result to the problem of recognizing even cycle matroids and to the problem of characterizing signed graphs with no odd-K_5 minor.

1 Introduction

In this article, we will consider graphs with multiple edges and loops. Let G be a graph. For a set $X \subseteq E(G)$, we write $V_G(X)$ to refer to the set of vertices incident to an edge of X and $G[X]$ for the subgraph with vertex set $V_G(X)$ and edge set X. A subset C of edges is a *cycle* if $G[C]$ is a graph where every vertex has even degree. An inclusion-wise minimal non-empty cycle is a *circuit*.

A *signed graph* is a pair (G, Σ) where G is a graph and $\Sigma \subseteq E(G)$. A subset $B \subseteq E(G)$ is *even* (resp. *odd*) if $|B \cap \Sigma|$ is even (resp. odd). In particular an edge e is odd if and only if $e \in \Sigma$. We say that Σ' is a *signature* of (G, Σ) if (G, Σ) and (G, Σ') have the same set of even cycles. Equivalently, Σ' is a signature of (G, Σ) if $\Sigma \triangle \Sigma'$ is a cut of G. We say that (G, Σ') is obtained from (G, Σ) by *resigning*. Given a graph H and $S \subseteq V(H)$, $\delta_H(S) := \{(u, v) \in E(H) : u \in S, v \notin S\}$ and we write $\delta_H(v)$ for $\delta_H(\{v\})$. [1] Let

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[1] Throughout the paper we shall omit indices when there is no ambiguity. For instance we may write $\delta(v)$ for $\delta_H(v)$.

(G, Σ) be a signed graph. A vertex s is a *blocking vertex* if for every odd circuit C, $s \in V(C)$. A pair of vertices s, t is a *blocking pair* if for every odd circuit C, $\{s, t\} \cap V(C) \neq \emptyset$. Denote by $\mathrm{loop}_G(u)$ the set of loops of G at vertex u. The set of all loops of G is denoted loop_G. It can be readily checked that s is a blocking vertex of (G, Σ) if and only if there exists a signature $\Sigma' \subseteq \delta_G(s) \cup \mathrm{loop}_G(s)$. Similarly, s, t is a blocking pair if and only if there exists a signature $\Sigma' \subseteq \delta_G(s) \cup \delta_G(t) \cup \mathrm{loop}_G(s) \cup \mathrm{loop}_G(t)$. Blocking pairs play an important role in a variety of problems on signed graphs as we will see in Section 2.

1.1 Displaying all blocking vertices

Let G be a graph with disjoint vertex sets A and B. An $A - B$ *path* is a path of H with one endpoint in A and one endpoint in B. We use "$a - b$ path" as shorthand for "$\{a\} - \{b\}$ path" and similarly, "$a - B$ path" as shorthand for "$\{a\} - B$ path". Let G be a graph with a vertex v and a set $\alpha \subseteq \delta_G(v)$ and suppose that $\mathrm{loop}_G(v) \cap \alpha = \emptyset$. We say that H is obtained from G by *splitting* v into v_1 and v_2 *according to* α if $V(H) = (V(G) \setminus \{v\}) \cup \{v_1, v_2\}$ and

$$\begin{aligned} E(H) = & \big\{(x, y) : (x, y) \in E(G) \setminus \delta_G(v)\big\} \cup \\ & \big\{(x, v_1) : (x, v) \in E(G) \cap \alpha\big\} \cup \big\{(x, v_2) : (x, v) \in E(G) \setminus \alpha\big\}. \end{aligned}$$

Consider now a signed graph (G, Σ) with a blocking vertex s. Suppose that we wish to describe the set of all blocking vertices of (G, Σ). We may assume that (G, Σ) has no odd loop Ω (for otherwise $\Omega \in \mathrm{loop}_G(s)$ and s is the unique blocking vertex). Thus there exists a signature $\Sigma' \subseteq \delta_G(s)$. Let H be obtained from G by splitting s into s_1 and s_2 according to Σ'. Observe now that there exists a bijection between odd circuits of (G, Σ) and $s_1 - s_2$ paths of H. It follows that the blocking vertices of (G, Σ) consist of s and vertices of H that are cut vertices separating s_1 and s_2. Thus if a signed graph has blocking vertices, these vertices can be displayed as special cut vertices in an auxiliary graph. In this paper we show analogous results for blocking pairs. Namely, we will show that if a signed graph has a blocking pair, then either it is special (i.e. is one of a number of well defined classes of signed graphs), or we can display every blocking pair as a special 2-separation in an auxiliary graph. We will get a number of different results depending on the connectivity conditions we consider.

1.2 Displaying blocking pairs using an auxiliary graph

In this section we show how to construct our auxiliary graph. We first need a number of definitions. For a set U of vertices, we denote by $G[U]$ the subgraph of G induced on the set U of vertices. We denote by $G - U$ the subgraph $G[V(G) \setminus U]$, and we use $G - v$ as shorthand notation for $G - \{v\}$. We define the *boundary* of X in G as $\mathcal{B}_G(X) := V_G(X) \cap V_G(\bar{X})$ where $\bar{X} := E(G) \setminus X$ and the *interior* of X in G as $\mathcal{I}_G(X) := V_G(X) \setminus \mathcal{B}_G(X)$. A *separator* in the graph G is a subset X of the edges which satisfies the property that $G[X]$ and $G[\bar{X}]$ are both connected and that $X, \bar{X}$ are non-empty. The *order* of a separator is given by $|\mathcal{B}_G(X)|$. A k*-separator* is a separator of order k. A k*-separation* is a separator X of order k where $|X|, |\bar{X}| \geq k$.

We say that $\vec{H}$ is an *LR-graph* if $\vec{H}$ is a graph with exactly two directed edges L and R and moreover, L and R are not loops. Consider a signed graph (G, Σ) with no even loops where $\Sigma \subseteq \delta(s) \cup \delta(t) \cup \text{loop}(s) \cup \text{loop}(t)$, for some $s, t \in V(G)$ (s, t is a blocking pair). Suppose first there are no edges in Σ with both endpoints in $\{s, t\}$ and construct an LR-graph $\vec{H}$ as follows:

- split s into s_1 and s_2 according to $\Sigma \cap \delta_G(s)$,
- split t into t_1 and t_2 according to $\Sigma \cap \delta_G(t)$,
- add directed edges $L = (s_1, t_1)$ and $R = (s_2, t_2)$.

Then we say that $\vec{H}$ is obtained by *unfolding* (G, Σ) on s, t and that (G, Σ) is obtained by *folding* $\vec{H}$.

Suppose now that there is an edge $f \in \Sigma$ with both endpoints in $\{s, t\}$. Then f behaves as if it consists of two series edges with exactly one in Σ. Namely, if f is an odd loop with endpoints s (resp. t), we add the edge (s_1, s_2) (resp. (t_1, t_2)); if f is an odd edge with endpoints s and t, we add the edge (s_1, t_2) or (s_2, t_1), chosen arbitrarily; (the choice depends on which of the two series edges used to represent f is in Σ). Suppose now there is an edge $f \notin \Sigma$ with endpoints in s and t. Then f behaves as it it consists of two series edges with the same parity. Namely, we add the edge (s_1, s_2) or (t_1, t_2), chosen arbitrarily; (the choice depends on whether both, or none of the two series edges used to represent f are in Σ).

For any vertex $v \in V(\vec{H})$, the *corresponding vertex* of v in G is defined as follows. If $v \notin \{s_1, s_2, t_1, t_2\}$, then v is a vertex of G and it is its own corresponding vertex. If $v \in \{s_1, s_2\}$, then s is the corresponding vertex to v, and similarly, if $v \in \{t_1, t_2\}$, then t is the corresponding vertex to v.

Remark 1. *Let (G, Σ) be a signed graph with a blocking pair s, t and a signature $\Sigma \subseteq \delta(s) \cup \delta(t) \cup \text{loop}(s) \cup \text{loop}(t)$. Let $\vec{H}$ be obtained by unfolding (G, Σ) on s, t. Let $X \subseteq E(\vec{H})$, where $L \in X$ and $R \notin X$. Given a vertex v of $\vec{H}$ denote by $\hat{v}$ the corresponding vertex of G.*

(1) If $\mathcal{B}_{\vec{H}}(X) = \{x\}$, then $\hat{x}$ is a blocking vertex of (G, Σ).

(2) If $\mathcal{B}_{\vec{H}}(X) = \{x, y\}$, then $\hat{x}, \hat{y}$ is a blocking pair of (G, Σ).

Proof. For (1) let $W = \{x\}$ and $\widehat{W} = \{\hat{x}\}$ for (2) let $W = \{x, y\}$ and $\widehat{W} = \{\hat{x}, \hat{y}\}$. Let C be an arbitrary odd circuit of (G, Σ). By the definition of $\vec{H}$, C is a path of $\vec{H}$ joining the tail of L and the tail of R or the head of L and the head of R. As $L \in X$ and $R \notin X$, we have $\emptyset \neq V_{\vec{H}}(C) \cap \mathcal{B}_{\vec{H}}(X) = V_{\vec{H}}(C) \cap W$. Hence, $\widehat{W} \cap V_G(C) \neq \emptyset$ and $\widehat{W}$ intersects every odd circuit of (G, Σ). $\square$

An *LR-separator* of $\vec{H}$ is a separator X of $\vec{H}$ of order 2, where $L \in X$ and $R \notin X$. We say that the LR-separator X in Remark 1(2) *displays* the blocking pair $\{\hat{x}, \hat{y}\}$.

1.3 The main results

In this paper we characterize when it is possible to unfold a signed graph and display all blocking pairs by LR-separators. We do this under various connectivity conditions. These connectivity conditions are motivated in Section 2.

1.3.1 3-connected even cycle matroid

We assume that the reader is familiar with the basics of matroid theory. See Oxley [14] for the definition of the terms used here. Let G be a graph. We denote by $\mathrm{cycle}(G)$ the set of all cycles of G. The set $\mathrm{cycle}(G)$ is the set of cycles of the *graphic* matroid of G. We identify $\mathrm{cycle}(G)$ with that matroid. We can extend this definition to a larger class of matroids. Let (G, Σ) be a signed graph. We denote by $\mathrm{ecycle}(G, \Sigma)$ the set of all even cycles of (G, Σ). The set $\mathrm{ecycle}(G, \Sigma)$ is the set of cycles of a binary matroid known as the *even-cycle matroid* [26]. We identify $\mathrm{ecycle}(G, \Sigma)$ with that matroid.

Theorem 2. *Let (G, Σ) be a signed graph such that $ecycle(G, \Sigma)$ is 3-connected and is not a graphic matroid. If (G, Σ) has at least one blocking pair, then exactly one of the following holds:*

(1) it can be unfolded such that every blocking pair can be displayed as an LR-separator, or

(2) it is an Octahedron, a Kite, a Saucer, or a Pinwheel.

We need to describe the terms Octahedron, Kite, Saucer and Pinwheel.

Before we can proceed we require a number of definitions. Let G be a graph and consider $X \subseteq E(G)$. We say that a path P is an $s - t|X$ path (or a path of *type* $s - t|X$) if $s, t \in V(X)$ and P is an $s - t$ path of $G[X]$ avoiding all vertices in $\mathcal{B}(X) \setminus \{s, t\}$. If $\mathcal{B}(X) = \{v_1, v_2, v_3\}$, $\mathcal{I}(X) = \{z\}$ and $E(X) = \{(z, v_1), (z, v_2), (z, v_3)\}$, then X is a *triad*. We say that X is *solid* if $G[X]$ is connected, there exists a circuit C in $G[X]$, $\mathcal{B}(X) = \{v_1, v_2, v_3\}$ and for $i = 1, 2, 3$ there exist $V(C) - v_i$ paths P_i of $G[X]$ where P_1, P_2, P_3 are vertex disjoint. (A path P_i in the previous definition may consists of a single vertex.) A *triangle* is a set of three edges that forms a circuit. Note that if X is a triangle of G then X is solid (in that case each of P_1, P_2, P_3 consists of a different vertex of the triangle X).

1.3.2 Octahedron

A signed graph (G, Σ) is an *Octahedron* if (after possibly resigning) there exist vertices a, b, c, d, s, t, and a partition A, B, C, D of $E(G)$ such that,

(i) $\mathcal{B}(A) = \{a, c, s\}, \mathcal{B}(B) = \{b, c, t\}, \mathcal{B}(C) = \{b, d, s\}, \mathcal{B}(D) = \{a, d, t\}$;

(ii) A, B, C, D are solid;

(iii) $\Sigma = \big(A \cap \delta(s)\big) \cup \big(D \cap \delta(t)\big)$.

An Octahedron is represented in Figure 1. A shaded region centered around a vertex indicates that every edge incident to that vertex that is in that region is odd. Kites come in three distinct flavors that we describe next.

1.3.3 Kite of Type I

A signed graph (G, Σ) is a *Kite of Type I* if (after possibly resigning) there exist vertices a, b, c, s, t, an edge $\Omega = (a, t)$, where $t \in \mathcal{B}(B)$ and a partition $A, B, C, \{\Omega\}$ of $E(G)$ such that,

(i) $\mathcal{B}(A) = \{a, b, s\}$, $\mathcal{B}(B) = \{b, c, t\}$, $\mathcal{B}(C) = \{a, c, s\}$;

(ii) A is solid or a triad and both B and C are solid;

(iii) $\Sigma = \big(A \cap \delta(s)\big) \cup \{\Omega\}$.

A Kite of Type I is represented in Figure 2. A shaded region centered around a vertex indicates that every edge incident to that vertex that is in that region is odd. The thick edge is also odd.

1.3.4 Kite of Type II

We say that (G, Σ) is a *Kite of Type II* if (after possibly resigning) there exist distinct vertices a, b, c, s, t, an edge $\Omega = (a, t)$ where $t \in \mathcal{B}(B)$ and a partition $A, B, C, \{\Omega\}$ of $E(G)$ such that,

(i) $\mathcal{B}(A) = \{a, b, s\}$, $\mathcal{B}(B) = \{a, b, c, t\}$, $\mathcal{B}(C) = \{a, c, s\}$;

(ii) There exist paths $P_1, \ldots, P_8$ of the following types:

$$
\begin{array}{llll}
P_1 : s - a|A & P_2 : s - b|A & P_3 : b - c|B & P_4 : t - b|B \\
P_5 : s - a|C & P_6 : s - c|C & P_7 : t - c|B & P_8 : t - a|B;
\end{array}
$$

(iii) $\Sigma = \big(\delta(s) \cap A\big) \cup \{\Omega\}$.

A Kite of Type II is represented in Figure 2.

1.3.5 Kite of Type III

A signed graph (G, Σ) is a *Kite of Type III* if (after possibly resigning) there exist vertices a, b, c, s, t, an edge $\Omega = (a, t)$, where $t \in \mathcal{B}(B)$ and a partition $A, B, C, D, \{\Omega\}$ of $E(G)$ such that,

(i) $\mathcal{B}(A) = \{a, b, s\}$, $\mathcal{B}(B) = \{b, c, t\}$, $\mathcal{B}(C) = \{a, c, s\}$, $\mathcal{B}(D) = \{a, b, c\}$.

(ii) There exist paths $P_1, \ldots, P_8$ of the following types:

$$
\begin{array}{llll}
P_1 : s - a|A & P_2 : s - b|A & P_4 : t - b|B & P_5 : s - a|C \\
P_6 : s - c|C & P_7 : t - c|B & &
\end{array}
$$

(iii) D is solid or a triad.

(iv) $\Sigma = \big(A \cap \delta(s)\big) \cup \{\Omega\}$.

A Kite of Type III is represented in Figure 2.

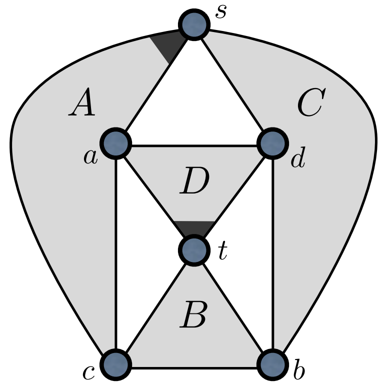
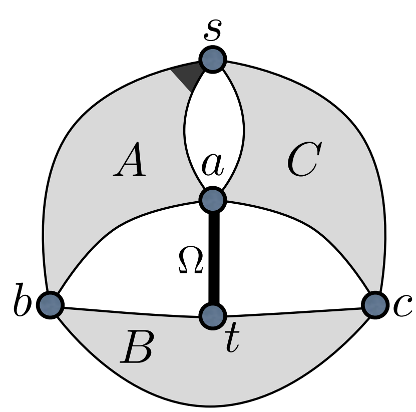
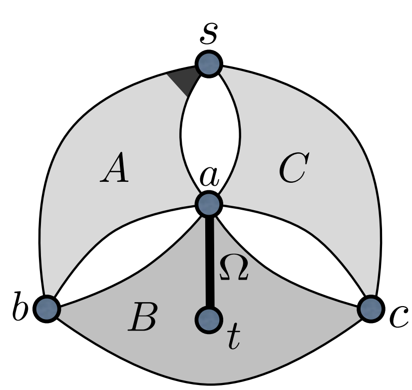


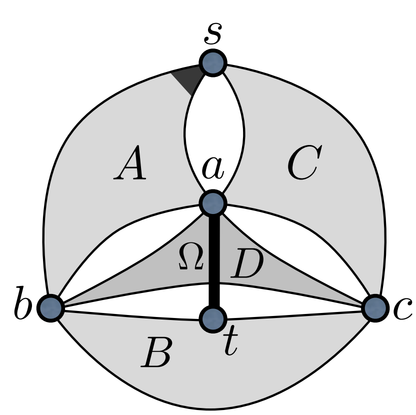
Figure 1: Octahedron.



Type I.



Type II.



Type III.

Figure 2: Kites.

1.3.6 Saucer

We say that (G, Σ) is a *Saucer* if (after possibly resigning) there exist distinct vertices a, b, c, d, s, t, an edge $\Omega = (a, t)$ where $t \in \mathcal{I}(D)$ and a partition $A_1, A_2, C_1, C_2, D, \{\Omega\}$ of $E(G)$ such that,

(i) $\mathcal{B}(A_1) = \{a, d, s\}$, $\mathcal{B}(A_2) = \{a, b, d\}$, $\mathcal{B}(C_1) = \{a, c, s\}$, $\mathcal{B}(C_2) = \{a, b, c\}$, $\mathcal{B}(D) = \{a, b\}$;

(ii) There exist paths $Q_1, \dots, Q_{10}$ of the following types:

$$\begin{array}{lllll} Q_1 : s - a|A_1 & Q_2 : s - d|A_1 & Q_3 : d - b|A_2 & Q_4 : a - d|A_1 \cup A_2 & Q_5 : s - a|C_1 \\ Q_6 : s - c|C_1 & Q_7 : c - b|C_2 & Q_8 : a - c|C_1 \cup C_2 & Q_9 : t - a|D & Q_{10} : t - b|D; \end{array}$$

(iii) $\Sigma = \big(\delta(s) \cap A_1\big) \cup \{\Omega\}$.

A Saucer is represented in Figure 3. A shaded region centered around a vertex indicates that, every edge incident to that vertex that is in that region, is odd. The thick edges is also odd.

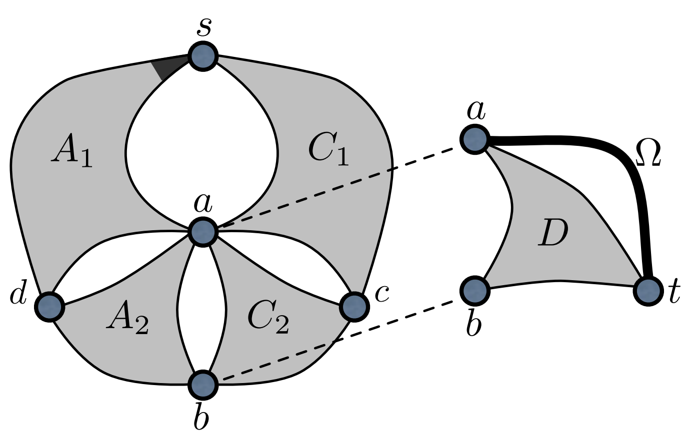


Figure 3: Saucer.

1.3.7 Pinwheel

Let H be a graph with a partition $B_1, \dots, B_r$ of $E(H)$ and distinct vertices $u_1, \dots, u_r$. Then H is a *flower* $\mathbb{F} := (B_1, \dots, B_r, u_1, \dots, u_r)$ if for all $i = 1, \dots, r$, $H[B_i]$ is connected and $\mathcal{B}_H(B_i) = \{u_i, u_{i+1}\}$ (where $r + 1 = 1$). We say that B_i is a *petal* of $\mathbb{F}$ and that u_i and u_{i+1} are the *attachments* of B_i. The flower $\mathbb{F}$ is *maximal* if no petal has a cut-vertex separating its attachments. Maximal flowers correspond to generalized circuits as introduced by Tutte in [21].

A signed graph (H, Γ) is an *odd flower* if H is a maximal flower and every odd circuit intersects every petal. Finally, a signed graph (G, Σ) is a *Whirligig* with *hub* $h \in V(G)$ if all its blocking pairs contain h, it has no blocking vertex, and the signed graph $(H, \Gamma) := \big(G - h, \Sigma \setminus \delta_G(h)\big)$ is an odd flower.

Remark 3. *Let (G, Σ) be a Whirligig with hub h. Then $H := G - h$ is a flower $\mathbb{F} = (B_1, \dots, B_r, u_1, \dots, u_r)$ and the following properties hold,* *(1) for every $i \in [r]$ there exists a signature Σ' of (G, Σ) where $\Sigma' \cap E(H) = \delta_H(u_i) \cap B_i$;*

(2) the blocking pairs of (G, Σ) are the sets $\{\{h, u_i\} : i \in [r]\}$.

Proof. **(1)** follows from the definition of odd flower. **(2)** because of (1) for every $i \in [r]$, $\{h, u_i\}$ is a blocking pair. By hypothesis, every blocking pair is of the form $\{h, u\}$. Suppose $u \neq u_i$ for every $i \in [r]$. Then $u \in \mathcal{I}_H(B_i)$ for some $i \in [r]$. It follows that u must be a cut vertex separating the attachments of B_i, contradicting the fact that H is maximal. $\square$

Let (G, Σ), and H be as in Remark 3. Consider a signature Σ' where $\Sigma' \cap E(H) = \delta_H(u_1) \cap B_1$. Let $\hat{\imath}$ is the largest index $i \in [r]$ for which there exists an edge $(h, w) \in \Sigma'$ where $w \in V_H(B_i) \setminus \{u_i, u_1\}$. Let $\hat{\jmath}$ is the smallest index $j \in [r]$ for which there exists an edge $(h, w) \in E(G) \setminus \Sigma'$ where $w \in V_H(B_j) \setminus \{u_{j+1}, u_1\}$. Note, that since (G, Σ') has no blocking vertex, (h, u_1) is not the only odd (resp. even) edge incident to the hub h. In particular, $\hat{\imath}, \hat{\jmath}$ are well defined. We say that (G, Σ) is 1-*degenerate* if either (a) there is no edge $(h, u_1) \in \Sigma'$ and $\hat{\imath} \leq \hat{\jmath}$ or (b) there is an edge $(h, u_1) \in \Sigma'$ and $\hat{\imath} = 1$ or $\hat{\jmath} = r$. Roughly speaking (G, Σ) is 1-degenerate if the edges incident to the hub are ordered such that, starting from u_1, all odd edges occur prior to the even edges. We define similarly what it means for the Whirligig to be k-degenerate for any $k \in [r]$ (relabel vertex u_k by u_1 and apply the previous definition). A *Pinwheel* is a Whirligig that is not k-degenerate for any $k \in [r]$.

Whirligigs are represented in Figure 4. A shaded region centered around a vertex indicates that, every edge incident to that vertex that is in that region, is odd. The thick edges are also odd. For the Whirligig on the left we have $\hat{\imath} = \hat{\jmath} = 2$, hence it is 1-degenerate.

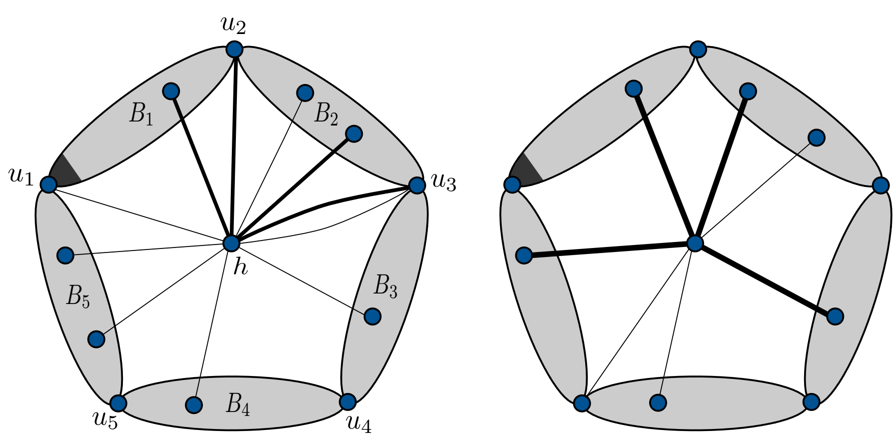


Figure 4: Left: 1-degenerate Whirligig. Right: Pinwheel.

1.3.8 Nearly 4-connected signed graphs

A signed graph that has no odd cycle is said to be *bipartite*. A separator X of a graph G is *trivial* if at least one of $\mathcal{I}(X)$ and $\mathcal{I}(\bar{X})$ is empty. We say that a signed graph (G, Σ) is *nearly* 4-*connected* if it satisfies the following conditions:

(a) $|\operatorname{loop}_G| \leq 1$ and if $e \in \operatorname{loop}_G$, then $e \in \Sigma$;

(b) $G \setminus \operatorname{loop}_G$ is 2-connected; [2]

(c) if G has a 2-separation X then (after possibly replacing X with $\bar{X}$) $(G[X], \Sigma \cap X)$ is non-bipartite and $\bar{X} = \{e, f\}$ where $|\{e, f\} \cap \Sigma| = 1$;

(d) if G has a non-trivial 3-separation X, then $(G[X], \Sigma \cap X)$ and $(G[\bar{X}], \Sigma \cap \bar{X})$ are non-bipartite.

An Octahedron is *trivial* if each of A, B, C, D form a triangle. A Kite is *basic* if it is a Kite of Type II and $\mathcal{I}(A) = \mathcal{I}(C) = \emptyset$.

Theorem 4. *Let (G, Σ) be a signed graph that is nearly 4-connected and has no blocking vertex. Suppose that no triangle is a signature. If (G, Σ) has at least one blocking pair, then exactly one of the following holds:*

(1) it can be unfolded such that every blocking pair can be displayed as an LR-separator, or

(2) it is a trivial Octahedron, a basic Kite, or a Pinwheel.

1.4 Outline of the paper

In Section 2 we discuss potential applications to Theorems 2 and 4. (Note, except for the definition of Lovász-flips, the material presented in that section is not required for the remainder of the paper.) It is shown in Section 3 that these theorems follow from two results, namely Theorem 12 and Proposition 13. The proof of the former is given in Section 4 while the latter result is proved in Section 5.

2 Applications

In this section, we illustrate how the study of blocking pairs plays a critical role in two open problems namely, the problem of recognizing even cycle matroids in polynomial time and the problem of characterizing signed graphs that are odd-K_5 free.

2.1 Recognizing even cycle matroids

If M is a binary matroid that is given by its $0, 1$ matrix representation, then it can be checked in polynomial time whether M is a graphic matroid [20, 16, 10, 22]. Zaslavsky [25, 26] introduced the class of signed graphic matroids. Pendavingh and Van Zwam [12] gave a recognition algorithm for the class of near-regular signed-graphic matroids. A recognition algorithm for the class of binary signed-graphic matroids is given in [11]. However, no such algorithm exists for the class of even cycle matroids. In this section we shall outline some of the challenges we face in finding such an algorithm and explain the relevance of Theorem 2.

[2]For a graph G and $D \subseteq E(G)$, $G \setminus D$ denotes the graph obtained from G by deleting edges D.

2.1.1 Representations of graphic matroids are nice

A graph G is a *representation* of a graphic matroid M if $M = \mathrm{cycle}(G)$. Consider a graph G and let $X \subseteq E(G)$. Suppose that $\mathcal{B}(X) = \{t_1, t_2\}$, for some $t_1, t_2 \in V(G)$. Let G' be obtained by identifying vertices t_1, t_2 of $G[X]$ with vertices t_2, t_1 of $G[\overline{X}]$ respectively. Then G' is obtained from G by a *Whitney-flip* on X. We will also call Whitney-flip the operation consisting of identifying two vertices from distinct components, or the operation consisting of partitioning the graph into components each of which is a block of G. Two graphs are *equivalent* if one can be obtained from the other by a sequence of Whitney-flips. As Whitney-flips preserve cycles, equivalent graphs are representations of the same graphic matroid. Whitney [24] proved the following seminal result.

Theorem 5. *All representations of a graphic matroid are equivalent.*

In particular, if a graphic matroid is 3-connected, it has a unique representation. This key property greatly facilitates the problem of recognizing graphic matroids.

2.1.2 Representations of even cycle matroids are naughty

A signed graph (G, Σ) is a *representation* of an even cycle matroids M if $M = \mathrm{ecycle}(G, \Sigma)$. Unfortunately, there is no simple description of the set of all representations of an even cycle matroid [8, 13]. Suppose that (G_1, Σ_1) and (G_2, Σ_2) are signed graphs where G_1 and G_2 are equivalent and Σ_2 is obtained from Σ_1 by resigning. Then (G_1, Σ_1) and (G_2, Σ_2) are *equivalent*. Equivalent signed graphs are representations of the same even cycle matroid. There is no analogue, for even cycle matroids, to Theorem 5 as the following result indicates.

Remark 6. *For any integer k there is a even cycle matroid M with $|E(M)| = 4k$ and 2^k pairwise inequivalent representations.*

We construct an example of such a matroid using the LR graphs introduced in the previous section.

Proof of Remark 6. Let $k \geq 1$ be any integer and let $\vec{H}$ be the LR-graph with $V(\vec{H}) = \{v_1, \ldots, v_{k+1}\} \cup \{v'_1, \ldots, v'_{k+1}\}$ and edges $L = (v_1, v'_1), R = (v_{k+1}, v'_{k+1})$ and $(v_i, v_{i+1}), (v'_i, v'_{i+1}), (v_i, v'_{i+1}), (v'_i, v_{i+1})$ for all $i \in [k]$. Let (G, Σ) be obtained by folding $\vec{H}$ and let $M = \mathrm{ecycle}(G, \Sigma)$. For any $j \in [k]$ define

$$X_j = \{L\} \cup \{(v_i, v_{i+1}), (v'_i, v'_{i+1}), (v_i, v'_{i+1}), (v'_i, v_{i+1}) : i = 1, \ldots, j-1\}.$$

Let $J \subseteq [k]$ and let $\vec{H}_J$ be obtained from $\vec{H}$ by doing a sequence of Whitney-flips on sets X_j for all $j \in J$ (since the sets X_j are nested it is easy to check that the order in which the Whitney-flips are done do not change the outcome). Let (G_J, Σ) be obtained by folding $\vec{H}_J$. (Note, Σ does not depend on the particular choice of J.) We leave it to the reader to verify that (G_J, Σ) is a representation of M and that for any pair $J_1, J_2 \subseteq [k]$ with $J_1 \neq J_2$ the corresponding signed graphs (G_{J_1}, Σ) and (G_{J_2}, Σ) are not equivalent. $\square$

If an even cycle matroid is graphic, then a complete description of its representations is known [3, 17]. Namely the following holds.

Theorem 7. *If an even cycle matroid is graphic, then any two or its representations are related by a sequence of Whitney-flips, resignings, and Lovász-flips.*

We include the definition of "Lovász-flip" here as it will arise again in Section 3. Consider a blocking pair v_1, v_2 of (G, Σ). We may assume, after possibly resigning, that $\Sigma \subseteq \delta(v_1) \cup \delta(v_2) \cup \mathrm{loop}(v_1) \cup \mathrm{loop}(v_2)$. We can construct a signed graph (G', Σ) from (G, Σ) by replacing the endpoints x, y of every odd edge e with the endpoints x', y' as follows:

- if $\{x, y\} = \{v_1, v_2\}$ then $x' = y'$ (i.e. e becomes a loop);
- if $x = y$ (i.e. e is a loop) then $x' = v_1$ and $y' = v_2$;
- if $x = v_i$ for some $i \in [2]$ and $y \neq v_1, v_2$, then $x' = v_{3-i}$ and $y' = y$.

In this case, we say that (G', Σ) is obtained from (G, Σ) by a *Lovász-flip* on v_1, v_2. It can be easily verified that Lovász-flips preserve even cycles. (Note, any two signed graphs (G_{J_1}, Σ) and (G_{J_2}, Σ) in Remark 6 are related by a sequence of Lovász-flips.)

2.1.3 Extending representations

Given a matroid M and $C, D \subseteq E(M)$, $N := M/C \setminus D$ denotes the matroid obtained by contracting elements C and deleting elements D. Then N is a *minor* of M and M is a *major* of N. We define minor operations on signed graphs next. Let (G, Σ) be a signed graph and let $e \in E(G)$. Then $(G, \Sigma) \setminus e$ is defined as $(G \setminus e, \Sigma - \{e\})$. We define $(G, \Sigma)/e$ as $(G \setminus e, \emptyset)$ if e is an odd loop of (G, Σ) and as $(G \setminus e, \Sigma)$ if e is an even loop of (G, Σ); otherwise $(G, \Sigma)/e$ is equal to $(G/e, \Sigma')$, [3] where Σ' is any signature of (G, Σ) which does not contain e. Consider $M = \mathrm{ecycle}(G, \Sigma)$ and disjoint sets $C, D \subseteq E(M)$. Let $N := M/C \setminus D$ and let $(H, \Gamma) := (G, \Sigma)/C \setminus D$. Then it can be readily checked that (H, Γ) is a representation of N. We say that (G, Σ) *extends* the representation (H, Γ) of N to the major M. The following is proved in [9].

Remark 8. *Suppose (G, Σ) has a blocking pair s_1, s_2 and $\Sigma \subseteq \delta(s_1) \cup \delta(s_2)$. For $i = 1, 2$, let H_i be obtained from G by splitting s_i into s_i', s_i'' according to $\Sigma \cap \delta_G(s_i)$ and adding an edge $\Omega = (s_i', s_i'')$. Then* $\mathrm{ecycle}(H_1, \Sigma) = \mathrm{ecycle}(H_2, \Sigma)$.

In the previous remark, let $N := \mathrm{ecycle}(G, \Sigma)$ and $M := \mathrm{ecycle}(H_1, \Sigma) = \mathrm{ecycle}(H_2, \Sigma)$. In general (H_1, Σ) and (H_2, Σ) need not be equivalent. Hence, a representation with a blocking pair of a matroid N may extend into several inequivalent representations of a major of M. Indeed, if M is a major of N, it is possible for M to have 2^k times as many inequivalent representation as N where $k = |E(M)| - |E(N)|$.

When we exclude blocking pairs however, the problem is better behaved as we explain next. An even cycle is *non-degenerate* if none of its representation has a blocking pair. The following was proved in [9].

[3] For a graph G and $D \subseteq E(G)$, G/D denotes the graph obtained from G by contracting edges D.

Theorem 9. *Let M be a 3-connected even cycle matroid which contains as a minor a non-degenerate 3-connected matroid N. Then the number of inequivalent representations of M is at most twice the number of inequivalent representations of N.*

Moreover, the proof of the previous theorem is constructive so that the representations of M can be constructed from the representations of N.

Suppose we are now given a 3-connected binary matroid M (given by a $0,1$ matrix) as well as a fixed size 3-connected non-degenerate minor N of M. Then we can check if M is an even cycle matroid as follows. Using Seymour's splitter theorem [15] we find a sequence of 3-connected matroids $N_0, N_1, \ldots, N_k$ with $N_0 = N$ and $N_k = M$ where, for any $i \in [k]$, $N_{i-1} = N_i/e$ or $N_{i-1} = N_i \setminus e$ for $e \in E(N_i) \setminus E(N_{i-1})$. Then we find the set of all inequivalent representations of N_0 (there are only a constant number of these since N_0 has fixed size). Finally, for all $i \in [k]$, we construct the representations of N_i from N_{i-1} (see Theorem 9). If any representation of $N = N_0$ extends to $M = N_k$ then M is an even cycle matroid, otherwise it is not.

As the example in Remark 6 illustrates, it is not possible to bound the number of inequivalent representations of an even cycle matroid. We can generalize the notion of equivalence, however, so that, all representations of the matroid in the proof of Remark 6 are "equivalent" under this new equivalence relation. We plan to prove an analogue to Theorem 9 with this new definition of equivalence (replacing, in the hypothesis, the condition that N be non-degenerate, with the condition that N be non-graphic). This would lead naturally to a recognition algorithm for even cycle matroids.

It is clear from Remark 8 however, that in order to understand the extensions of a representation (G, Σ) of a matroid N to a major M we need a complete understanding of the blocking pairs of (G, Σ). Theorem 2 is precisely the tool that is required.

2.2 Structure of odd-K_5 free signed graphs

Graphs without K_4 minors are series parallel graphs. Wagner [23] showed that graphs without K_5 minors can be constructed by pasting planar graphs and one special graph along edges and triangles. It is natural to try to extend these results to signed graphs. An *odd-K_n* is the signed graph $\big(K_n, E(K_n)\big)$. A signed graph is *odd-K_n free* if it does not have an odd-K_n minor. Gerards [5] gave a structural characterization of odd-K_4 free signed graphs. Recently Conforti and Gerards gave a structure theorem for a subclass of signed graphs without odd-K_5 minors [1]. No structural characterization exists for the class of all odd-K_5 free signed graphs, however. These signed graphs play an important role in multi-commodity flow problems [7, 4]. We wish to outline how Theorem 4 is relevant to the study of this class of signed graphs.

The following are basic classes of odd-K_5-free signed graphs (G, Σ) [6]:

(B1) G is planar;

(B2) (G, Σ) has a blocking pair;

(B3) (G, Σ) has an even-face embedding on the double pinched sphere;

(B4) (G, Σ) has an even-face embedding on the pinched projective plane;

(B5) (G, Σ) has an even-face embedding on the Klein bottle.

There are other basic classes that we omit here in the interest of brevity.

We can define decompositions operations for a signed graph (G, Σ) (analogous to the operations in Wagner [23] theorem) with the property that (G, Σ) is odd-K_5 free if and only if each of its parts is odd-K_5 free. A signed graph is *irreducible* if cannot be decomposed. It can be shown that irreducible signed graphs are nearly 4-connected. [4] We wish to prove that every irreducible odd-K_5 free signed graph is in a basic class or belongs to a "thin" (highly structured) class of signed graphs that we can fully describe. A set of signed graphs $\mathcal{U}$ is *unavoidable* if every odd-K_5 free signed graph that is irreducible but not basic has a minor in $\mathcal{U}$. A general proof strategy is to find an unavoidable set $\mathcal{U}$ and then for each $(H, \Gamma) \in \mathcal{U}$ prove the conjecture for the signed graphs with a minor (H, Γ). The success of such a strategy hinges on our ability to find such a set $\mathcal{U}$ where none of the signed graphs are in a basic class.

As a proof of concept, let us sketch a strategy for finding an unavoidable set $\mathcal{U}$ where none of the signed graphs are in (B1) or (B2). Kuratowski's theorem says that every graph that is not planar must either contain K_5 or $K_{3,3}$ as a minor. Thus, if we let $\mathcal{U}_1$ be the set of all signed graphs that are of the form (K_5, Σ) (and not equivalent to odd-K_5) and of the form $(K_{3,3}, \Sigma)$, then $\mathcal{U}_1$ is unavoidable. Clearly, no signed graph $(H, \Gamma) \in \mathcal{U}_1$ is in (B1), but (H, Γ) may have a blocking pair, i.e. may be in (B2).

We require some definition and a conjecture to proceed further. Let $\mathcal{F}$ be a set of signed graphs and let (G, Σ) be a signed graph. Then (H, Γ) is an *$\mathcal{F}$-minor* of (G, Σ) if it is a minor of (G, Σ) that has a minor in $\mathcal{F}$. We say that (G, Σ) is *minimally blocking-pair free* with respect to $\mathcal{F}$ if (G, Σ) has a minor in $\mathcal{F}$, it has no blocking pair, it is nearly 4-connected, and every $\mathcal{F}$-minor (H, Γ) that is nearly 4-connected has a blocking pair.

Conjecture 10. *Let $\mathcal{F}$ be a finite set of nearly 4-connected signed graphs. Let $\mathcal{F}'$ be the set of signed graphs that are minimally blocking-pair free with respect to $\mathcal{F}$. Then $\mathcal{F}'$ is finite, moreover, we can find an explicit description of $\mathcal{F}'$ from $\mathcal{F}$.*

Thus if the conjecture holds, we can construct a finite set $\mathcal{U}_2$ that is minimally blocking-pair free with respect to $\mathcal{U}_1$. Then $\mathcal{U}_2$ is an unavoidable set and no signed graph in $\mathcal{U}_2$ is in either (B1) or (B2).

Theorem 4 states that if a nearly 4-connected signed graph (H, Γ) has a blocking pair and it is not one of two special families of signed graphs (or a trivial Octahedron), then either Γ is a triangle or we can represent every blocking pair as a 2-separation in an auxiliary graph with labeled vertices. Thus, the problem of finding signed graphs (G, Σ) which contain (H, Γ) as a minor but do not have blocking pairs themselves reduces to finding graphs containing the auxiliary graph as a minor and which do not have certain 2-separations. There are standard techniques, known as *blocking sequences*, which allow one to

[4] In this particular decomposition scheme we are not attempting to bound the number of parts of the decomposition, otherwise a slightly less restrictive connectivity condition needs to be considered.

find graphs containing a fixed minor which do not have certain separations. See [2] for more details. We conclude that Theorem 4 offers a tool for proving Conjecture 10 and hence, for the study of signed graphs with no odd-K_5 minor.

3 A generalization

3.1 Nearly 3-connected signed graphs

We first find a common relaxation to the notion of 3-connected even cycle matroids and nearly 4-connected signed graphs. Recall that a signed graph is bipartite if it has no odd cycles. A signed graph (G, Σ) is *nearly* 3-*connected* if it satisfies the following conditions:

(a) $|\,\mathrm{loop}_G\,| \leq 1$ and if $e \in \mathrm{loop}_G$, then $e \in \Sigma$;

(b) $G \setminus \mathrm{loop}_G$ is 2-connected;

(c) if G has a non-trivial 2-separation X, then $(G[X], \Sigma \cap X)$ and $(G[\bar{X}], \Sigma \cap \bar{X})$ are non-bipartite.

Clearly, every nearly 4-connected signed graph is nearly 3-connected. In [13] it is shown,

Proposition 11. *If* $\mathrm{ecycle}(G, \Sigma)$ *is* 3*-connected, then* (G, Σ) *is nearly* 3*-connected.*

3.2 Statement of the main results

We present two results on nearly 3-connected signed graphs in this subsection and explain how these imply Theorems 2 and 4. Let us call a signed graph (G, Σ) *timid* if it has at least one blocking pair and it cannot be unfolded such that every blocking pair can be displayed as an LR-separator.

A signed graph (G, Σ) is a *Shredder* if there exists distinct vertices x_1, x_2, x_3, a signature Σ' and a partition C_0, C_1, C_2, C_3 of $E(G)$ (where C_i is possibly empty for some values of i) such that:

(i) $\mathcal{B}(C_i) \subseteq \{x_1, x_2, x_3\}$ for all $i \in \{0\} \cup [3]$, and

(ii) $\Sigma' \cap C_i = \delta(x_i) \cap C_i$ for all $i \in [3]$ and $\Sigma' \cap C_0 = \emptyset$.

A Shredder is represented in Figure 5. The vertices on the dotted lines are to be identified. A shaded region centered around a vertex indicates that every edge incident to that vertex that is in that region is odd. We now state the two key results of the paper.

Theorem 12. *Let* (G, Σ) *be a nearly* 3*-connected signed graph that has no blocking vertex. If* (G, Σ) *is timid, then it is either a Shredder, an Octahedron, a Kite, a Saucer, or a Pinwheel.*

Proposition 13. *Nearly* 3*-connected Octahedrons, Kites, Saucers, and Pinwheels are timid.*

We distinguish two cases for Theorem 12. The case where any two blocking pairs of (G, Σ) share a vertex is proved in Section 3.5 (Proposition 17). The case where there exists disjoint blocking pairs for (G, Σ) is addressed in Section 4 (Theorem 20). Finally, the proof of Proposition 13 is given in Section 5.

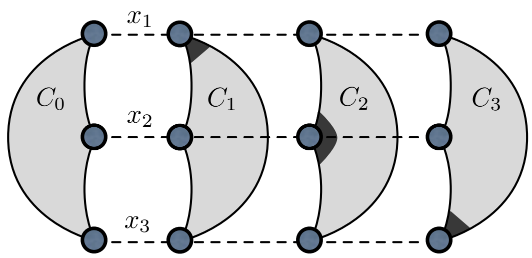


Figure 5: Shredder

3.3 A corollary: Theorem 2

We require the following observation,

Remark 14. *If the signed graph* (G, Σ) *has a blocking vertex, then* $\text{ecycle}(G, \Sigma)$ *is graphic.*

Proof. Let s denote the blocking vertex. Then there exists a signature Σ' where $\Sigma' \subseteq \delta_G(s) \cup \text{loop}_G(s)$. Let H be obtained from G by splitting v into v_1, v_2 according to Σ' (and replacing any edge in $\text{loop}_G(s) \cap \Sigma'$ by a (v_1, v_2) edge). Then $\text{ecycle}(G, \Sigma) = \text{cycle}(H)$. □

Lemma 15. *If* (G, Σ) *is a Shredder, then* $\text{ecycle}(G, \Sigma)$ *is graphic.*

Proof. Let x_1, x_2, x_3, C_0, C_1, C_2, C_3 and Σ' as given in the definition of Shredder. We may assume that C_1 has no edges with both endpoints in $\{x_1, x_2, x_3\}$ as we can redefine C_0, C_2, C_3 so as to contain these edges. Let B denote the cut $\delta_G(V(C_1) \setminus \{x_1, x_2, x_3\})$. Let Γ denote the signature $\Sigma' \triangle B$. Then

$$\Gamma = \big[(\delta_G(x_2) \cup \delta_G(x_3)) \cap C_1\big] \cup \big[\delta_G(x_2) \cap C_2\big] \cup \big[\delta_G(x_3) \cap C_3\big].$$

Hence, all odd edges have an endpoint in $\{x_2, x_3\}$. Let (H, Γ) be the signed graph obtained from (G, Γ) by a Lovász-flip on x_2, x_3. Observe that both C_2 and C_3 are 2-separators of H. Let H' be obtained from H by a Whitney-flip on both C_2 and C_3. Note that (H, Γ) has a blocking vertex v. As Lovász-flips preserve even cycles, $\text{ecycle}(G, \Gamma) = \text{ecycle}(H, \Gamma)$. The result now follows from Remark 14. □

Proof of Theorem 2. Since $\text{ecycle}(G, \Sigma)$ is 3-connected, it follows by Proposition 11 that (G, Σ) is nearly 3-connected. As (G, Σ) is not graphic, Remark 14 implies that (G, Σ) has no blocking vertex. Hence, (G, Σ) satisfies the hypothesis of Theorem 12. Suppose (1) does not hold, i.e. (G, Σ) is timid. Together with Lemma 15 it implies that (2) holds. Suppose that (2) holds. Then Proposition 13 implies that (G, Σ) is timid, i.e. (1) does not hold. □

3.4 A corollary: Theorem 4

We require the following observations.

Remark 16. *Let* (G, Σ) *be nearly* 4*-connected.*

(1) *If (G, Σ) is a Shredder, then Σ is contained in a triangle.*

(2) *If (G, Σ) is an Octahedron, then it is a trivial Octahedron.*

(3) *If (G, Σ) is a Kite, then it is a basic Kite.*

(4) *(G, Σ) is not a Saucer.*

Proof. In this proof (c),(d) refer to the conditions of nearly 4-connected signed graphs (see Section 1.3.8). **(1)** Let x_1, x_2, x_3, C_0, C_1, C_2, C_3 and Σ' as given in the definition of Shredder. If $V(G) = \{x_1, x_2, x_3\}$ the result is trivial. Thus we may assume, after possibly resigning and relabeling the sets X_i, that $\mathcal{I}(X_0) \neq \emptyset$. It follows from conditions (c) and (d) that $\mathcal{I}(X_i) = \emptyset$ for $i = 1, 2, 3$. Hence, all edges of $X_1 \cup X_2 \cup X_3$ have both ends in $\{x_1, x_2, x_3\}$. **(2)** Since A, B, C, D are solid, it follows from conditions (d) that each of A, B, C, D are triangles. **(3)** If (G, Σ) is a Kite of Type I or of Type III, then the separation $B \cup \{\Omega\}$ contradicts condition (d). Condition (d) also implies that $\mathcal{I}(A) = \mathcal{I}(C) = \emptyset$. **(4)** If (G, Σ) is a Saucer, then the separation $D \cup \{\Omega\}$ contradicts (c). □

Proof of Theorem 4. Since (G, Σ) is nearly 4-connected, it is nearly 3-connected. Suppose (1) does not hold, i.e. (G, Σ) is timid. Then by Theorem 12 and Remark 16, (2) must hold. Suppose that (2) holds. Then Proposition 13 implies that (1) does not hold. □

3.5 The case without disjoint blocking pairs

We prove the following result in this section.

Proposition 17. *Let (G, Σ) be a nearly 3-connected signed graph that has no blocking vertex. If (G, Σ) is timid and has no two disjoint blocking pairs, then it is either a Shredder or a Pinwheel.*

A *blocking pair triple* in a signed graph (G, Σ) is a set of three distinct vertices x_1, x_2, x_3 such that each of $\{x_1, x_2\}$, $\{x_2, x_3\}$ and $\{x_1, x_3\}$ are blocking pairs.

Lemma 18. *A signed graph without blocking vertices and with a blocking pair triple is a Shredder.*

Proof. Suppose that (G, Σ) has a blocking pair triple x_1, x_2, x_3. Let $k \geq 1$ be a positive integer, and let $Z_1, \ldots, Z_k$ be the components of $G - \{x_1, x_2, x_3\}$. Pick a signature Σ' such that $\Sigma' \subseteq \delta(x_1) \cup \delta(x_2) \cup \delta(x_3)$. For every $j \in [k]$, we let B_j be the set of edges $\delta(V(Z_j)) \cup E(Z_j)$, i.e. the edges of Z_j along with the edges with one endpoint in $V(Z_j)$ and the other endpoint contained in $\{x_1, x_2, x_3\}$. For every $i \in [3]$ and $j \in [k]$, the edge set $\delta(x_i) \cap B_j$ is either entirely contained in Σ' or is disjoint from Σ', lest there exists an odd cycle contained in B_j which avoids vertices of $\{x_1, x_2, x_3\} \setminus \{x_i\}$, contradicting the fact that $\{x_1, x_2, x_3\} \setminus \{x_i\}$ is a blocking pair. Thus, we may assume (by possibly considering the signature $\Sigma' \triangle \delta(V(Z_j))$, for $j \in [k]$) that for all indices $j \in [k]$, the set $(B_j \cap \delta(x_i)) \cap \Sigma' \neq \emptyset$ for at most one index $i \in [3]$. Let X be the set of edges with both endpoints in $\{x_1, x_2, x_3\}$. Suppose first that $X = \emptyset$. Then the statement now follows if we let $C_0 = \{B_j : j \in [k] \text{ and } B_j \cap \Sigma = \emptyset\}$ and for $i \in [3]$, we let $C_i = \{B_j : j \in [k] \text{ and } B_j \cap \Sigma \subseteq \delta(x_i)\}$. If $X \neq \emptyset$, add all even edges of X to C_0, add odd edges (x_1, x_2) and (x_1, x_3) of X to B_1 and add odd edge (x_2, x_3) of X to B_2. Then the statement still holds. □

Lemma 19. *Let (G, Σ) be a nearly 3-connected signed graph that has no blocking vertex. If (G, Σ) is timid and all blocking pairs use the same vertex, then it is a Pinwheel.*

Proof. Let (G, Σ) denote a timid signed graph where the set of all blocking pairs is of the form $\big\{\{h, u_i\} : i \in [r]\big\}$. Since (G, Σ) is timid, $r \geq 2$. Define $(H, \Gamma) := \big(G - h, \Sigma \setminus \delta(h)\big)$. Then $u_1, \ldots, u_r$ are blocking vertices of (H, Γ). We may assume, after possibly resigning, that $\Gamma \subseteq \delta_H(u_1)$. Let H' be obtained from H by splitting u_1 into vertices u_1^- and u_1^+ according to Γ. Then (see Section 1.1) the vertices $u_2, \ldots, u_r$ are exactly the cut vertices of H' separating u_1^- and u_1^+. It follows that H is a maximal flower $\mathbb{F} = (B_1, \ldots, B_r, u_1, \ldots, u_r)$ for some petals $B_1, \ldots, B_r$ and that (H, Γ) is an odd flower. In particular, (G, Σ) is a Whirligig with hub h.

Suppose for a contradiction that (G, Σ) is not a Pinwheel, i.e. it is k-degenerate for some $k \in [r]$. We may assume (after possibly relabeling the petals and attachments) that it is 1-degenerate. Let Σ', $\hat{\imath}$, $\hat{\jmath}$ be as in the definition of 1-degenerate Whirligig given in Section 1.3.7. Then either

(a) there is no edge $(h, u_1) \in \Sigma'$ and $\hat{\imath} \leq \hat{\jmath}$ or

(b) there is an edge $(h, u_1) \in \Sigma'$ and $\hat{\imath} = 1$ or $\hat{\jmath} = r$.

Let $\vec{H}$ be the LR-graph obtained from (G, Σ') by unfolding on h, u_1. We will show that every blocking pair h, u_i can be displayed as an LR-separation of $\vec{H}$ thereby contradicting the fact that (G, Σ) is timid. Denote by h^- (resp. h^+) the tail of L (resp. R) of $\vec{H}$ and denote by v_1^- (resp. v_1^+) the head of L (resp. R) of $\vec{H}$. For $i \in [r]$, define,

$$Y_i := B_1 \cup \ldots \cup B_{i-1}$$
$$X_i := \{L\} \cup Y_i \cup \{e \in E(\vec{H}) : e = (h^-, w) \text{ or } e = (h^+, w) \text{ where } w \in V_H(Y_i)\}.$$

Note, that $L \in X_i$ and $R \notin X_i$. Consider $i \in [r]$. By the definition of unfolding, if there exists $e = (h, v_1) \in E(G) \setminus \Sigma'$ then either $e = (h^-, v_1^-)$ or $e = (h^+, v_1^+)$ in $\vec{H}$. Suppose first that (a) occurs. Then by construction, for all $i \in [r]$ where $i < \hat{\jmath}$ we have that $\mathcal{B}_{\vec{H}}(X_i) = (h^-, u_i)$, and for all $i \in [r]$where $i \geq \hat{\jmath}$ we have that $\mathcal{B}_{\vec{H}}(X_i) = (h^+, u_i)$. Hence, blocking pairs of (G, Σ) are displayed as LR-separators. Suppose now that (b) occurs. We consider the case where $\hat{\imath} = 1$ only as the case where $\hat{\jmath} = r$ is similar. By the definition of unfolding we can choose the edge $e = (h, u_1) \in \Sigma'$ of G to have endpoints h^+, v_1^- in $\vec{H}$. Then (h, u_1) is displayed by the LR-separator, $\{L\}$ and by construction, for all $i \in \{2, \ldots, r\}$, $\mathcal{B}_{\vec{H}}(X_i) = (h^+, u_i)$. □

Proof of Proposition 17. Construct an auxiliary graph H with $V(H) = V(G)$ as follows: $(u, v) \in E(H)$ if and only if $\{u, v\}$ is a blocking pair of (G, Σ). By hypothesis, $E(H) \neq \emptyset$ and H has no two independent edges. It follows that H is a triangle or H is a star. In the former case, Lemma 18 implies that (G, Σ) is a Shredder. In the latter case, Lemma 19 implies that (G, Σ) is a Pinwheel. □

3.6 Organization of the remainder of the paper

Theorem 12 will follow from Proposition 17 and the following result.

Theorem 20. *Let (G,Σ) be a nearly 3-connected signed graph that has no blocking vertex. If (G,Σ) is timid and has two disjoint blocking pairs, then it is either a Shredder, an Octahedron, a Kite, or a Saucer.*

Theorem 20 and Proposition 13 are proved in Section 4 and in Section 5 respectively.

4 The proof of Theorem 20

We say that a signed graph is *relevant* if it satisfies the following properties:

(h1) it is nearly 3-connected;

(h2) it has no blocking vertex;

(h3) it has two disjoint blocking pairs;

(h4) it has no blocking triple;

(h5) it is timid.

Because of (h1) a relevant signed graph has no even loops and because of (h3) it has no odd loops either. If (G,Σ) satisfies the hypothesis of Theorem 20 it satisfies conditions (h1), (h2), (h3) and (h5). If in addition, it is not a Shredder, then by Lemma 18 it also satisfies (h4), hence it is relevant. Therefore,

Remark 21. *To prove Theorem 20 it suffices to show that a relevant signed graph is either an Octahedron, a Kite, or a Saucer.*

4.1 Sketch of the proof

We given an overview of the proof in this section.

4.1.1 U-graphs

A *U-graph* is a pair $(H,\mathcal{U})$ where H is a graph, and $\mathcal{U}$ is an ordered set of four distinct vertices. Consider an LR graph $\vec{H}$. We can construct a U-graph $(H,\mathcal{U})$ from $\vec{H}$ as follows: $H = \vec{H} \setminus \{L, R\}$ and the first vertex of $\mathcal{U}$ is the tail of L, the second vertex of $\mathcal{U}$ is the head of L, the third vertex of $\mathcal{U}$ is the tail of R, and finally the fourth vertex of $\mathcal{U}$ is the head of R. Note, that the constructing is reversible, i.e. given the U-graph $(H,\mathcal{U})$ we can construct the LR-graph $\vec{H}$. We say that $\vec{H}$ is the LR-graph *corresponding* to the U-graph $(H,\mathcal{U})$ and that $(H,\mathcal{U})$ is the U-graph corresponding to the LR-graph $\vec{H}$. An LR-graph $\vec{H}$ *arises* from a signed graph (G,Σ) if $\vec{H}$ is obtained by unfolding (G,Σ') on s,t for some signature Σ' and some pair of vertices s,t of G.

Suppose $\vec{H}$ is an LR-graph and let $(H,\mathcal{U})$ be the corresponding U-graph. We say that $(H,\mathcal{U})$ is obtained by *unfolding* (G,Σ) on s,t if $\vec{H}$ is obtained by unfolding (G,Σ) on s,t. We also say that the U-graph $(H,\mathcal{U})$ *arises* from (G,Σ) if $\vec{H}$ arises from (G,Σ). Finally, if for some $X \subseteq E(H)$, a blocking pair $\{a,b\}$ of (G,Σ) is displayed by an LR-separator $X \cup \{L\}$ of $\vec{H}$, then we say that $\{a,b\}$ is *displayed* by X in the U-graph $(H,\mathcal{U})$.

Consider a U-graph $(H,\mathcal{U})$ where $\mathcal{U} = (s_1,t_1,s_2,t_2)$. [5] We say that $(H,\mathcal{U}')$ is equivalent to $(H,\mathcal{U})$ if either, (i) $\mathcal{U}' = (t_1,s_1,t_2,s_2)$, or (ii) $\mathcal{U}' = (s_2,t_2,s_1,t_1)$, or (iii) $\mathcal{U}' = (t_2,t_1,s_2,s_1)$. The following remark shows that if a U-graph arises from a signed graph (G,Σ) then so does every equivalent U-graph.

Remark 22. *Consider the U-graphs $(H,\mathcal{U})$ and $(H,\mathcal{U}')$ given in the previous definition. Suppose that $(H,\mathcal{U})$ is obtained by unfolding a signed graph (G,Σ) on some vertices s,t. In case (i), $(H,\mathcal{U}')$ is obtained by unfolding (G,Σ) on t,s.* [6] *Let $\Sigma' = \Sigma \triangle \delta_G(s) \triangle \delta_G(t)$. In case (ii), $(H,\mathcal{U}')$ is obtained by unfolding (G,Σ') on s,t. In case (iii), $(H,\mathcal{U}')$ is obtained by unfolding (G,Σ') on t,s,*

4.1.2 Intercepting sets

Let $(H,\mathcal{U})$ be a U-graph with $\mathcal{U} = (s_1,t_1,s_2,t_2)$. Distinct vertices $a,b \in V(H) \setminus \mathcal{U}$ form an *intercepting pair* of $(H,\mathcal{U})$ if $H - \{a,b\}$ has no $s_1 - s_2$ path and no $t_1 - t_2$ path. Distinct vertices a,b,c, where $a \in V(H) \setminus \mathcal{U}$ and either $\{b,c\} = \{s_1,s_2\}$ or $\{b,c\} = \{t_1,t_2\}$, form an *intercepting triple* of $(H,\mathcal{U})$ if $H - \{a,b,c\}$ has no $s_1 - s_2$ path and no $t_1 - t_2$ path. An *intercepting set* is either an intercepting pair or an intercepting triple.

Lemma 23. *Let $(H,\mathcal{U})$ be a U-graph with $\mathcal{U} = (s_1,t_1,s_2,t_2)$ obtained by unfolding a signed graph (G,Σ) on s,t. Consider a blocking pair $\{a,b\} \neq \{s,t\}$ of (G,Σ). Let A (resp. B) be the set of vertices of H corresponding to vertex a (resp. b) of G. If $\{a,b\} \cap \{s,t\} = \emptyset$, then $A \cup B$ is an intercepting pair of H; otherwise $A \cup B$ is an intercepting triple of $(H,\mathcal{U})$.*

Proof. By hypothesis $\{a,b\} \neq \{s,t\}$. If $(A \cup B) \cap \mathcal{U} = \emptyset$, then $A = \{a\}$ and $B = \{b\}$. Otherwise, $A \cap \mathcal{U} \neq \emptyset$, and we may assume, $A = \{a\}$ and either $b = s$ or $b = t$. In the former case $B = \{s_1,s_2\}$, in the latter case $B = \{t_1,t_2\}$. Let C be an arbitrary $s_1 - t_1$ or $s_2 - t_2$ path of H. Then C is an odd circuit of (G,Σ). It follows that $V_G(C) \cap \{a,b\} \neq \emptyset$. Hence, $V_H(C) \cap (A \cup B) \neq \emptyset$. □

Let $(H,\mathcal{U})$ be a U-graph and let $\vec{H}$ be the corresponding LR-graph. We say that an intercepting set W of $(H,\mathcal{U})$ is *good* if we can display the blocking pair corresponding to W in the U-graph $(H,\mathcal{U})$. Equivalently, W is good if for $\vec{H}$, the LR-graph corresponding to $(H,\mathcal{U})$, and for some $X \subseteq E(H)$, we have that $X \cup \{L\}$ is an LR-separator and that $\mathcal{B}_{\vec{H}}(X \cup \{L\}) \subseteq W$.

Lemma 24. *Every U-graph, that arises from a relevant signed graph, has a bad intercepting set.*

[5] We use the notation $(v_1,\ldots,v_k)$ to denote an ordered sequence $v_1,\ldots,v_k$ of vertices.

[6] By definition of unfolding, if $\vec{H}$ is obtained by unfolding (G,Σ) on s,t and $\vec{H}'$ is obtained by unfolding (G,Σ) on t,s, then $\vec{H}'$ is obtained from $\vec{H}$ by reversing the direction of both L and R.

Proof. Let $(H,\mathcal{U})$ be a U-graph with $\mathcal{U} = (s_1, t_1, s_2, t_2)$ that is obtained from a relevant signed graph (G, Σ) by unfolding on some vertices s, t. Suppose for contradiction that every intercepting set is good. Let $\{a, b\}$ be an arbitrary blocking pair of (G, Σ). By Lemma 23, either $\{a, b\} = \{s, t\}$ or $\{a, b\}$ corresponds to an intercepting set W of $(H,\mathcal{U})$. Since W is good it can be displayed in $(H,\mathcal{U})$. As $\{a, b\}$ was arbitrary, (G, Σ) is not timid, contradicting (h5). $\square$

4.1.3 Templates

Let $(H,\mathcal{U})$ be a U-graph and let $\vec{H}$ be the corresponding LR-graph. Let $\{a, b\}$ be a (good) intercepting pair of $(H,\mathcal{U})$, that is displayed by a set $X \subseteq E(H)$. We say that $\{a, b\}$ is *skewed*, if there exist edges e, f of $E(H)$ such that $\{e, f\}$ is an edge cut of $\vec{H}$ separating L from R, $e \in X$, $f \notin X$ and where a is an endpoint of e and b is an endpoint of f (or vice-versa).

Let $(H,\mathcal{U})$ be a U-graph with $\mathcal{U} = (s_1, t_1, s_2, t_2)$. Let $\{a, b\}$ be a (good) intercepting pair of $(H,\mathcal{U})$, that is displayed by a set $X \subseteq E(H)$. Let E_0 be the set of edges with both ends contained in either: $\{s_1, t_1\}$, $\{s_2, t_2\}$, or $\{a, b\}$. We say that $\mathbb{T} = (H,\mathcal{U}, \{a, b\}, X)$ is a *template* if the following conditions hold.

(T1) $H[X \setminus E_0]$ and $H[\bar{X} \setminus E_0]$ are both connected;

(T2) there is no vertex $z_i \in \{s_i, t_i\}$, for both $i = 1, 2$, with both z_1 and z_2 of degree one in H;

(T3) $\{a, b\}$ are not skewed.

We omit the set X in the 4-tuple $\mathbb{T} = (H,\mathcal{U}, \{a, b\}, X)$, when it is not relevant. Let $\mathbb{T} = (H,\mathcal{U}, \{a, b\})$ be a template and let $(H,\mathcal{U}')$ be equivalent to $(H,\mathcal{U})$. Then observe that $\mathbb{T}' := (H,\mathcal{U}', \{a, b\})$ is also a template. We say that $\mathbb{T}$ and $\mathbb{T}'$ are *equivalent*. A template $\mathbb{T} = (H,\mathcal{U}, \{a, b\})$ *arises* from a signed graph (G, Σ) if the U-graph $(H,\mathcal{U})$ arises from (G, Σ).

The first key step of the proof is the following result.

Lemma 25. *For every relevant signed graph (G, Σ) there is a template that arises from (G, Σ).*

The proof is postponed until Section 4.3

4.1.4 Breaking the argument into different cases

Let us first classify intercepting sets.

Remark 26. *Let $\mathbb{T} = (H,\mathcal{U}, \{a, b\})$ be a template where $\mathcal{U} = (s_1, t_1, s_2, t_2)$ and let W be an intercepting set of $(U,\mathcal{U})$ different from $\{a, b\}$. Then, after possibly replacing $\mathbb{T}$ by an equivalent template, W is of one of the following types,*

Type A. $W = \{x, y\}$ *and* $x, y \notin \{a, b\} \cup \mathcal{U}$.

Type B. $W = \{x, s_1, s_2\}$, *and* $x \notin \{a, b\} \cup \mathcal{U}$.

Type C. $W = \{x, y\}$, $x \in \{a, b\}$ *and* $y \notin \{a, b\} \cup \mathcal{U}$.

Type D. $W = \{x, s_1, s_2\}$ *and* $x \in \{a, b\}$.

The following result will be proved in Section 4.2.

Lemma 27. *No template has a bad intercepting set of Type D.*

Consider a template $\mathbb{T} = (H, \mathcal{U}, \{a, b\}, X)$ where $\mathcal{U} = (s_1, t_1, s_2, t_2)$. We say that $\mathbb{T}$ has an *internal pivot* if there is no $s_1 - t_1$ path in $H[X] - \{a, b\}$ and no $s_2 - t_2$ path in $H[\bar{X}] - \{a, b\}$. We say that $\mathbb{T}$ has an *external pivot* if there is no $a - b$ path in $H[X] - \{s_1, t_1\}$ and no $a - b$ path in $H[\bar{X}] - \{s_2, t_2\}$.

Let us now classify templates.

Remark 28. *Let* $\mathbb{T} = \big(H, \mathcal{U}, \{a, b\}\big)$ *where* $\mathcal{U} = (s_1, t_1, s_2, t_2)$*, be a template that arises from a relevant signed graph. Then, after possibly replacing* $\mathbb{T}$ *by an equivalent template,* $\mathbb{T}$ *is of one of the following types,*

Type 1. $\mathbb{T}$ *has a bad intercepting set of Type A,*

Type 2. $\mathbb{T}$ *has a bad intercepting set of Type B and has an external pivot,*

Type 3. $\mathbb{T}$ *has a bad intercepting set of Type C and has an internal pivot,*

Type 4. $\mathbb{T}$ *has a bad intercepting set of Type B and has no external pivot,*

Type 5. $\mathbb{T}$ *has a bad intercepting set of Type C and has no internal pivot.*

Proof. We know from Lemma 24 that there exists a bad intercepting set W of $\mathbb{T}$. Remark 26 implies that W is Type A,B,C or D. Lemma 27 implies that W is not of Type D. The result now easily follows. $\square$

Let (G, Σ) be a relevant signed graph and let $\mathbb{T}$ be a template that arises from (G, Σ). By the previous remark, $\mathbb{T}$ is of Type i, for some $i \in [5]$. We say that $\mathbb{T}$ is i*-extremal* for (G, Σ) if there is no template $\mathbb{T}'$ [7] arising from (G, Σ) of Type j, where $j > i$. In Section 4.2 we will show the following result.

Lemma 29. *Let* (G, Σ) *be a signed graph and let* $\mathbb{T}$ *be a template arising from* (G, Σ)*.*

(1) If $\mathbb{T}$ *is of Type 2, then there is a template* $\mathbb{T}'$ *arising from* (G, Σ) *of Type 3.*

(2) If $\mathbb{T}$ *is of Type 4, then there is a template* $\mathbb{T}'$ *arising from* (G, Σ) *of Type 5.*

Next we state the three key lemmas of the proof.

Lemma 30. *If a 1-extremal template* $\mathbb{T}$ *arises from a relevant signed graph* (G, Σ)*, then* (G, Σ) *is an Octahedron.*

[7] where $\mathbb{T}'$ is possibly equal to $\mathbb{T}$, as a template can be of more than one type (depending on the intercepting set considered).

Lemma 31. *If a 3-extremal template $\mathbb{T}$ arises from a relevant signed graph (G, Σ), then (G, Σ) is an Saucer.*

Lemma 32. *If a 5-extremal template $\mathbb{T}$ arises from a relevant signed graph (G, Σ), then (G, Σ) is a Kite.*

Proof of Theorem 20. Let (G, Σ) be a relevant signed graph. As indicated in Remark 21 it suffices to show that (G, Σ) is an Octahedron, a Kite, or a Saucer. Lemma 25 implies that there exists a template that arises from (G, Σ). Hence, by Remark 28, there exists an i-extremal template $\mathbb{T}$, that arises from (G, Σ), for some $i \in [5]$. Lemma 29 implies that $i \notin \{2, 4\}$. Hence, $\mathbb{T}$ is 1-, 3-, or 5-extremal. Now the result follows immediately from lemmas 30, 31 and 32. $\square$

The remainder of this section is organized as follows. Section 4.2 proves Lemma 29 and Lemma 27 and establishes some definitions and results that are used throughout the rest of the paper. Section 4.3 proves Lemma 25. Lemma 30, 31, 32 are proved in Sections 4.4, 4.6, 4.5 respectively.

4.2 Preliminaries

First, we prove connectivity properties of U-graphs arising from relevant signed graphs.

Lemma 33. *Let $(H, \mathcal{U})$ be a U-graph with $\mathcal{U} = (s_1, t_1, s_2, t_2)$, that arises from a signed graph (G, Σ) that is nearly 3-connected and has no blocking vertex. Suppose Y is a separator of H or order k. Then $k \geq 1$. If $k = 1$, then Y or $\bar{Y}$ consists of a single edge, say Ω, and the end of Ω that is not in $\mathcal{B}_H(Y)$ is in $\mathcal{U}$. If $k = 2$ and $|Y|, |\bar{Y}| \geq 2$, then $\mathcal{U} \cap \mathcal{I}_H(Y), \mathcal{U} \cap \mathcal{I}_H(\bar{Y}) \neq \emptyset$.*

Proof. Consider the case where $k = 0$. Since, by (h1), (G, Σ) is nearly 3-connected, G is 2-connected. Hence, for some $i, j \in [2]$, $s_i \in \mathcal{I}_H(Y)$, $s_{(3-i)} \in \mathcal{I}_H(\bar{Y})$ and $t_j \in \mathcal{I}_H(Y)$, $t_{(3-j)} \in \mathcal{I}_H(\bar{Y})$. It follows that (G, Σ) is bipartite, contradicting (h2).

Consider the case where $k = 1$ and denote by z the unique vertex in $\mathcal{B}_H(Y)$. Suppose first that $\mathcal{B}_G(Y) \geq 3$. Then, for some $i, j \in [2]$, $s_i \in \mathcal{I}_H(Y)$, $s_{(3-i)} \in \mathcal{I}_H(\bar{Y})$ and $t_j \in \mathcal{I}_H(Y)$, $t_{(3-j)} \in \mathcal{I}_H(\bar{Y})$. It follows that z is a blocking vertex of (G, Σ), contradicting (h2). Thus $\mathcal{B}_G(Y) \leq 2$ and since (G, Σ) is nearly 3-connected $|\mathcal{B}_G(Y)| = 2$. Suppose $|Y|, |\bar{Y}| \geq 2$. As (G, Σ) is nearly 3-connected, $(G[Y], \Sigma \cap Y)$ and $(G[\bar{Y}], \Sigma \cap \bar{Y})$ are non-bipartite. Then, (after possibly exchanging the role of Y and $\bar{Y}$) we have that $s_1, s_2 \in V_H(Y)$ and $t_1, t_2 \in V_H(\bar{Y})$. But then $\mathcal{B}_G(Y) = 1$, a contradiction. Thus Y or $\bar{Y}$ consists of a single edge Ω. As G is 2-connected, the end of Ω that is not in $\mathcal{B}_H(Y)$ is in $\mathcal{U}$.

Consider the case where $k = 2$ and $|Y|, |\bar{Y}| \geq 2$. Suppose for a contradiction that $\mathcal{U} \cap \mathcal{I}_H(Y) = \emptyset$. Then either $\mathcal{U} \cap \mathcal{B}_H(Y)$ is equal to one of $\{s_1, s_2\}$, $\{t_1, t_2\}$ and Y is a 1-separation of G, or otherwise Y is a 2-separation of G and $(G[Y], \Sigma \cap Y)$ is bipartite. In both cases this contradicts the fact that (G, Σ) is nearly 3-connected. Thus $\mathcal{U} \cap \mathcal{I}_H(Y) \neq \emptyset$ and similarly, $\mathcal{U} \cap \mathcal{I}_H(\bar{Y}) \neq \emptyset$. $\square$

Next, we establish properties of bad intercepting sets.

Lemma 34. *Let $(H, \mathcal{U})$ be a U-graph, with $\mathcal{U} = (s_1, t_1, s_2, t_2)$, obtained by unfolding a relevant signed graph (G, Σ) on s, t and let W be a bad intercepting set of $(H, \mathcal{U})$.*

(1) If $W = \{a, b\}$ ($a, b \notin \mathcal{U}$), then, for some $i \in [2]$, there exists an $s_i - t_{(3-i)}$ path in $H - \{a, b\}$.

(2) If $W = \{s_1, s_2, a\}$ ($a \notin \mathcal{U}$), then, for all $i \in [2]$, there exists a $t_i - s_{(3-i)}$ path of $H - \{a, s_i, t_{(3-i)}\}$.

Proof. Let $\vec{H}$ be the LR-graph corresponding to $(H, \mathcal{U})$. **(1)** Let Z denote the edges in the component of $\vec{H} - \{a, b\}$ that contains s_1 and t_1 (s_1, t_1 are in the same component because of edge L). Since $\{a, b\}$ is an intercepting pair, $H - \{a, b\}$ contains no $s_1 - t_1$ and no $s_2 - t_2$ paths. Suppose for a contradiction that (1) does not hold. Then $s_2, t_2 \notin V_{\vec{H}}(Z)$. Define $X := Z \cup \{(a, z) \in E(\vec{H}) : z \in V_{\vec{H}}(Z)\} \cup \{(b, z) \in E(\vec{H}) : z \in V_{\vec{H}}(Z)\}$. By construction $\vec{H}[X]$ is connected and $\mathcal{B}_{\vec{H}}(X) \subseteq \{a, b\}$. Moreover, the last inclusion is not strict, for otherwise Remark 1(1) implies that (G, Σ) has a blocking vertex, contradicting (h2). Lemma 33 implies that $\vec{H}[\bar{X}]$ is connected. Hence, X is an LR-separator displaying $\{a, b\}$, a contradiction as W is bad. **(2)** We assume $i = 1$ as the case $i = 2$ is similar. Let Z denote the edges in the component of $\vec{H} - \{a, s_1\}$ that contains t_1. Suppose for a contradiction $H - \{a, s_1\}$ has no $t_1 - \{s_2, t_2\}$ path. Then $s_2, t_2 \notin V_{\vec{H}}(Z)$. Define $X := Z \cup \{(s_1, z) \in E(\vec{H}) : z \in V_{\vec{H}}(Z)\} \cup \{(a, z) \in E(\vec{H}) : z \in V_{\vec{H}}(Z)\}$. Proceeding as in case (1) we deduce that X is an LR-separator displaying $\{s, a\}$, a contradiction as W is bad. Thus $H - \{a, s_1\}$ has either (i) a $t_1 - s_2$ path Q or (ii) a $t_1 - t_2$ path Q. Among all paths Q satisfying (i) or (ii) pick one with minimum number of edges. If (i) occurs, then we are done. If (ii) occurs, then Q is a path of $H - \{a, s_1, s_2\}$, contradicting the fact that W is an intercepting set. $\square$

Proof of Lemma 27. We may assume, after possibly exchanging the labels of a and b, that $W = \{a, s_1, s_2\}$. Lemma 34(2) implies that $H - \{a, s_1, t_2\}$ has a $t_1 - s_2$ path. In particular, there exists a $t_1 - b$ path P in $H[X] - \{a, s_1\}$. Similarly, we show that there exists a $t_2 - b$ path Q in $H[\bar{X}] - \{a, s_2\}$. But then $P \cup Q$ is a $t_1 - t_2$ path of $H - \{a, s_1, s_2\}$ contradicting the fact that W in an intercepting pair. $\square$

Let $\big(H, (s_1, t_1, s_2, t_2)\big)$ be a U-graph and let $\vec{H}$ be the corresponding LR-graph. Suppose that for some $X \subseteq E(H)$, $X \cup \{L\}$ is an LR-separator where $\mathcal{B}(X \cup \{L\}) = \{a, b\}$ for some $a, b \in V(H)$. Let H_1 denote the subgraph of H with edges $\bar{X}$ and vertices $V_H(\bar{X}) \cup \{a, b\}$. Let H_2 denote the subgraph of H with edges X and vertices $V_H(X) \cup \{a, b\}$. For $i = 1, 2$ denote by a_i (resp. b_i) the vertex of H_i corresponding to a (resp. b). Let H' be obtained from H_1 and H_2 by identifying vertices s_1 with s_2 (calling the resulting vertex s) and by identifying t_1 with t_2 (calling the resulting vertex t). We say that the U-graph $\big(H', (a_1, b_1, a_2, b_2)\big)$ is obtained from $\big(H, (s_1, t_1, s_2, t_2)\big)$ by *shifting* X, see Figure 6. Note, a, b need not be disjoint from s_1, t_1, s_2, t_2. (For instance if for $i \in [2]$, $s_i = a$ in H then $a_{(3-i)} = s$ in H'.)

Remark 35. *Suppose that a U-graph $(H', \mathcal{U}')$ is obtained by shifting a U-graph $(H, \mathcal{U})$. Then $(H, \mathcal{U})$ and $(H', \mathcal{U}')$ arise from the same signed graph.*

Proof. Assume that $H, H', \mathcal{U}, \mathcal{U}', a, b, X$ are as in the definition of shifting. Suppose that $(H, \mathcal{U})$ is obtained by unfolding a signed graph (G, Σ) on some vertices s, t. We only consider the case where a, b are disjoint from s_1, s_2, t_1, t_2 as the other cases are similar. Let $\Sigma' := \Sigma \triangle \delta_G(X) \triangle \delta_G(s) \triangle \delta_G(t)$. Note, Σ' is a signature of (G, Σ). It can be readily checked that $(H', \mathcal{U}')$ is obtained by unfolding (G, Σ') on a, b. $\square$

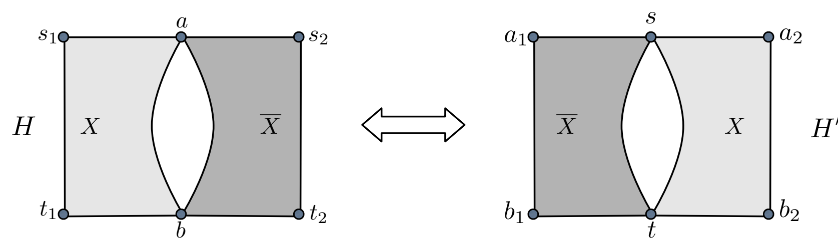


Figure 6: Shift

Given a path P of a graph G and vertices $u, v \in V(P)$ we denote by $P[u, v]$ the subpath of P with ends u, v.

Proof of Lemma 29. Let $\mathbb{T} = \big(H,\mathcal{U}, \{a, b\}, X\big)$ be a template where $\mathcal{U} = (s_1, t_1, s_2, t_2)$. Let $\big(H',\mathcal{U}'\big)$ where $\mathcal{U}' = (a_1, b_1, a_2, b_2)$ be the U-graph obtained from $(H,\mathcal{U})$ by shifting X. By Remark 35, $(H',\mathcal{U}')$ also arises from (G, Σ).

Claim 1. $\mathbb{T}' = \big(H',\mathcal{U}', \{s, t\}, \bar{X}\big)$ *is a template.*

Proof. The sets E_0 are the same for both $\mathbb{T}$ and $\mathbb{T}'$ hence, property (T1) for $\mathbb{T}$ implies (T1) for $\mathbb{T}'$. Property (T2) for $\mathbb{T}$ implies property (T3) for $\mathbb{T}'$ and property (T3) for $\mathbb{T}$ implies property (T2) for $\mathbb{T}'$. $\diamond$

Since $\mathbb{T}$ is of Type 2 or of Type 4 there exists a bad intercepting pair W of $(H,\mathcal{U})$ of Type B, i.e. $W = \{x, s_1, s_2\}$, $x \notin \{a, b\} \cup \mathcal{U}$ and there is no $s_1 - s_2$ and no $t_1 - t_2$ path of $H - \{x, s_1, s_2\}$.

Claim 2. $W' := \{x, s\}$ *is a bad intercepting pair of* $(H',\mathcal{U}')$.

Proof. We claim that W' is an intercepting pair of $(H',\mathcal{U}')$. Let P be an arbitrary $a_1 - a_2$ path of H'. If P uses s in H', then P is an $s_1 - s_2$ path of H. [8] If P uses t in H', then P is a $t_1 - t_2$ path of H. In either cases, $V_H(P) \cap W \neq \emptyset$, and hence, $V_{H'}(P) \cap W' \neq \emptyset$. Hence, $H' - W'$ has no $a_1 - a_2$ path and similarly, $H' - W'$ has no $b_1 - b_2$ path. Therefore, W' is an intercepting pair of $\mathbb{T}'$ of Type C.

It remains to show that W' is bad. Since W is bad, Lemma 34 implies that there exists a $t_1 - s_2$ path P_1 of $H - \{s_1, x, t_2\}$ and a $t_2 - s_1$ path P_2 of $H - \{s_2, x, t_1\}$. Consider first the case where P_1 uses a and P_2 uses b. Let Q be the set of edges of H in $P_1[t_1, a] \cup P_2[b, t_2]$. Then Q is a $b_1 - a_2$ path of $H' - \{x, s\}$. In particular, x, s is not a good intercepting set of $(H',\mathcal{U}')$. Otherwise we may assume, after possibly interchanging the labels of a, b, that both P_1 and P_2 use a. But then $P_1[t_1, a] \cup P_2[a, t_2]$ is a path of $H - \{s_1, s_2, x\}$, contradicting the fact that W is an intercepting set of $(H,\mathcal{U})$. $\diamond$

Finally, note that $\mathbb{T}$ has an external pivot, if and only if $\mathbb{T}'$ has an internal pivot. Hence, if $\mathbb{T}$ is of Type 2 then $\mathbb{T}'$ is of Type 3 and if $\mathbb{T}$ is of Type 4 then $\mathbb{T}'$ is of Type 5. $\square$

We will also require the following connectivity result,

[8] We identify paths with their set of edges.

Lemma 36. *Let (G, Σ) be nearly 3-connected. Let Y be a separator of G with $\mathcal{B}(Y) = \{v_1, v_2, v_3\}$. Suppose that $(G[Y], \Sigma \cap Y)$ is bipartite and that for all distinct $i, j, k \in [3]$ there exists a $v_i - v_j$ path in $G[Y] - v_k$. Then Y is either solid or a triad.*

Proof. If $G[Y]$ has a circuit C then, as $(G[Y], \Sigma \cap Y)$ is bipartite and since (G, Σ) is nearly 3-connected, there exist three, vertex disjoint, $V(C) - \{v_1, v_2, v_3\}$ paths of $G[Y]$. Hence, Y is solid. Thus, we may assume that $\mathcal{I}(Y) \neq \emptyset$ (for otherwise Y is a triangle), and that $G[Y]$ is acyclic. As (G, Σ) is nearly 3-connected, $\mathcal{I}(Y)$ consists of a single vertex z and Y is the triad $(z, v_1), (z, v_2), (z, v_3)$. $\square$

4.3 Proof of Lemma 4.3

The goal of this section is to show that a template arises from every relevant signed graph. Before we prove this result we shall require some preliminaries.

Let (G, Σ) be a signed graph. We say that a pair $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of blocking pairs are *twins* if there exist edges e, f of G such that $\{e, f\}$ is a signature of (G, Σ), and for all $i \in [2]$, exactly one of x_i, y_i is an end of e and exactly one of x_i, y_i is an end of f. Note that, in this definition we do not require $\{x_1, y_1\}$ and $\{x_2, y_2\}$ to be disjoint

Lemma 37. *A relevant signed graph (G, Σ) has a pair of disjoint blocking pairs that are not twins.*

Proof. We will assume, for a contradiction, that any two pairs of blocking pairs are twins. By (h3) there exist disjoint blocking pairs, say $\{a_1, b_1\}$ and $\{a_2, b_2\}$. Hence, (after possibly exchanging the labels of a_1 and b_1) there exist edges $e = (a_1, a_2)$ and $f = (b_1, b_2)$ such that $\{e, f\}$ is a signature.

Claim 1. *We may assume, after possibly relabeling a_1, a_2, b_1, b_2, that there exists a blocking pair $\{a_2, y\}$ where $y \notin \{a_1, a_2, b_1, b_2\}$.*

Proof. There exists a blocking pair $\{x, y\}$ distinct from $\{a_i, b_j\}$ for all $i, j \in [2]$, for otherwise the LR-graph $\vec{H}$ obtained by unfolding $(G, \{e, f\})$ on a_1, b_1 displays all blocking pairs, contradicting (h5). Let $k := |\{x, y\} \cap \{a_1, a_2, b_1, b_2\}|$. Note, if $k = 1$ we are done. Suppose $k = 2$, then $\{x, y\} = \{a_1, a_2\}$ or $\{x, y\} = \{b_1, b_2\}$ and (G, Σ) has a blocking triple, contradicting (h4). Thus $k = 0$. Since $\{a_1, a_2\}$ and $\{x, y\}$ are twins, it follows that $\{a_2, x\}$ or $\{a_2, y\}$ is a blocking pair. $\diamond$

Since $\{a_1, b_2\}$ and $\{a_2, y\}$ are twins, either,

(a) there exist edges $g_1 = (a_1, a_2)$ and $g_2 = (b_2, y)$ such that $\{g_1, g_2\}$ is a signature, or

(b) there exist edges $g_3 = (a_1, y)$ and $g_4 = (a_2, b_2)$ such that $\{g_3, g_4\}$ is a signature.

Since $\{a_1, b_1\}$ and $\{a_2, y\}$ are twins, either,

(c) there exist edges $g_5 = (a_1, a_2)$ and $g_6 = (b_1, y)$ such that $\{g_5, g_6\}$ is a signature, or

(d) there exist edges $g_7 = (a_1, y)$ and $g_8 = (a_2, b_1)$ such that $\{g_7, g_8\}$ is a signature.

Claim 2. *In (a) $g_1 \neq e$ and in (c) $g_5 \neq e$.*

Proof. Suppose for a contradiction that $g_1 = e$. Since $\{e, f\}$ and $\{g_1, g_2\}$ are both signatures, $\{e, f\} \triangle \{g_1, g_2\} = \{f, g_2\}$ is a cut. Hence, either $\deg(b_2) = 2$ or b_2 is a cut vertex, a contradiction to (h1). The case $g_5 = e$ is similar. $\diamond$

Suppose (a) and (c) occur. By Claim 2, g_1 and g_5 are distinct from e. Since $\{e, f\}$ is a signature of (G, Σ) edges g_1 and g_5 have the same parity. It follows from (h1) that $g_1 = g_5$. Since $\{g_1, g_2\}$ and $\{g_5, g_6\}$ are signatures, $\{g_2, g_6\}$ is a cut. Hence, either $\deg(y) = 2$ or y is a cut vertex, a contradiction to (h1). Suppose (a) and (d) occur. Since no two parallel edges of (G, Σ) have the same parity (and by Claim 2, $e \neq g_1$) $\{e, g_1\}$ is an odd circuit, a contradiction as $\{g_7, g_8\}$ is a signature. The case where (b) and (c) occurs, is symmetric to the previous one, so it does not occur either. Suppose (b) and (d) occur. Consider first the case where $g_3 = g_7$. Since $\{g_3, g_4\}$ and $\{g_7, g_8\}$ are signatures, $\{g_4, g_8\}$ is a cut. Hence, either $\deg(a_2) = 2$ or a_2 is a cut vertex, a contradiction to (h1). Consider now the case where $g_3 \neq g_7$. Since no two parallel edges of (G, Σ) have the same parity $\{g_3, g_7\}$ is an odd circuit, a contradiction as $\{e, f\}$ is a signature. $\square$

Remark 38. *If a U-graph $\big(H, (s_1, t_1, s_2, t_2)\big)$ arises from a signed graph (G, Σ) then so does the U-graph $(H, \mathcal{U}')$ where either (i) $\mathcal{U}' = (s_2, t_1, s_1, t_2)$, or (ii) $\mathcal{U}' = (H, (s_1, t_2, s_2, t_1)$.*

Proof. $\big(H, (s_1, t_1, s_2, t_2)\big)$ is obtained by unfolding (G, Σ') on some vertices s, t where Σ' is a signature of (G, Σ). In case (i), $(H, \mathcal{U}')$ is obtained by unfolding (G, Γ) where $\Gamma = \Sigma' \triangle \delta_G(s)$. In case (ii), $(H, \mathcal{U}')$ is obtained by unfolding (G, Γ) where $\Gamma = \Sigma' \triangle \delta_G(t)$. $\square$

Lemma 39. *If $\{a, b\}$ is a bad intercepting pair of the U-graph $\big(H, (s_1, t_1, s_2, t_2)\big)$ then, $\{a, b\}$ is a good intercepting pair of the U-graph $\big(H, (s_2, t_1, s_1, t_2)\big)$.*

Proof. Lemma 34(1) implies that for some $i \in [2]$, there is an $s_i - t_{(3-i)}$ path in $H - \{a, b\}$. If there is an $s_1 - t_1$ path Q in $H - \{a, b\}$, then $P \cup Q$ contains an $s_1 - s_2$ or a $t_1 - t_2$ path in $H - \{a, b\}$, contradicting the fact that $\{a, b\}$ is an intercepting pair. Hence, there is no $s_1 - t_1$ and (by a similar argument) no $s_2 - t_2$ path in $H - \{a, b\}$. This implies that $\{a, b\}$ is not a bad intercepting pair of $\big(H, (s_2, t_1, s_1, t_2)\big)$ for otherwise Lemma 34(1) would imply that there exists a path Q in $H - \{a, b\}$ with ends corresponding to either the first and fourth element of (s_2, t_1, s_1, t_2) or the second and third element of (s_2, t_1, s_1, t_2), i.e. that Q is an $s_2 - t_2$ path or an $s_1 - t_1$ path, a contradiction. $\square$

Consider a graph H with $V(H) = \{s_1, t_1, s_2, t_2, a, a', b, b'\}$ and

$$E(H) := \{(s_1, t_1), (a', b), (a, b'), (s_1, a'), (a', a), (a, s_2), (t_1, b), (b, b'), (b', t_2)\}.$$

We say that a U-graph equivalent to $(H, \mathcal{U})$, where $\mathcal{U} = (s_1, t_1, s_2, t_2)$ or $\mathcal{U} = (s_1, t_1, t_2, s_2)$, is a *Ladder*. (See Figure 7.) We say that a signed graph (G, Σ) is a Ladder if it is obtained by folding a U-graph $(H, \mathcal{U})$ that is a Ladder. It can be readily checked that $(H, \mathcal{U})$ displays all the blocking pairs of (G, Σ). Hence,

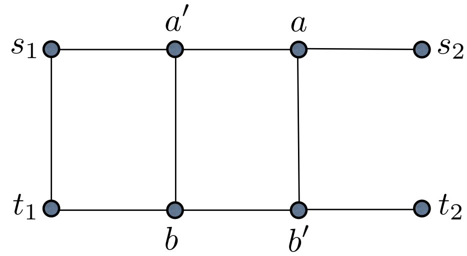


Figure 7: Ladder

Remark 40. *If a signed graph is a Ladder, then it is not timid.*

Consider a graph H with a set of distinct vertices $s_1, t_1, s_2, t_2, a, a', b$ and for which we can partition $E(H)$ into edges, $(a', b), (s_1, a'), (a', a), (t_1, b), (b, t_2)$ and a set B where $\mathcal{B}(B) = \{a, t_2\}$, $s_2 \in \mathcal{I}(B)$. We say that the U-graph equivalent to $(H,\mathcal{U})$, where $\mathcal{U} = (s_1, t_1, s_2, t_2)$ or $\mathcal{U} = (s_1, t_1, t_2, s_2)$, is a *Widget*. (See Figure 8.) We say that a signed graph is a Widget if it is obtained by folding a U-graph that is a Widget.

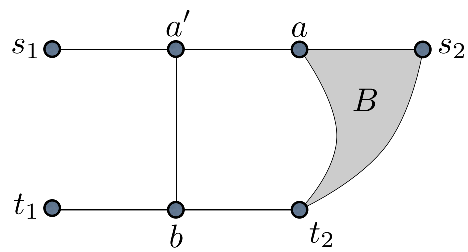


Figure 8: Widget

Lemma 41. *If a signed graph (G, Σ) is a relevant Widget, then there is a template that arises from (G, Σ).*

Proof. Suppose H is as in the definition of Widget. Consider first the case where $\mathcal{U} = (s_1, t_1, t_2, s_2)$. Denote by x (resp. y) the vertex of G obtained by identifying vertices t_1 and s_2 (resp. t_2 and s_1) of H. Then $\{a', b, y\}$ is a blocking triple of (G, Σ) contradicting (h3). Thus we may assume that $\mathcal{U} = (s_1, t_1, s_2, t_2)$. Let s (resp. t) denote the vertex of G obtained by identifying vertices s_1 and s_2 (resp. t_1 and t_2) of H. All of the following blocking pairs are displayed in the U-graph $(H,\mathcal{U})$: $\{s, t\}, \{a', b\}, \{a, t\}, \{s, b\}, \{a', t\}, \{a, b\}$. Because of (h5), there exists another blocking pair $\{x, y\}$. Since (G, Σ) has a pair of odd and even edges in between vertices b, t, we may assume that $y \in \{b, t\}$ and $x \notin \{a, a', s\}$, hence $x \in \mathcal{I}_G(B)$. Note, however that if $\{b, x\}$ is a blocking pair, then so is $\{t, x\}$. $H - \{t_2, x\}$ has no $s_1 - s_2$ path, for otherwise $\{t_1, t_2, x\}$ is not an intercepting set of H, contradicting Remark 23. It follows that there is a partition B_1, B_2 of B such that $V_H(B_1) \cap V_H(B_2) = \{t_2, x\}$, $a \in \mathcal{I}_H(B_1)$ and $s_2 \in \mathcal{I}_H(B_2)$. Among all such partition B_1, B_2 of B, pick one where B_1 is inclusion-wise minimal. Let $(H',\mathcal{U}')$ be obtained from $(H,\mathcal{U})$ by shifting $\{(s_1, a'), (t_1, b)\}$. Let t denote the vertex of H' corresponding to vertices t_1 and t_2 of H. By Remark 35 $(H',\mathcal{U}')$ arises from (G,Σ). Then $\mathbb{T} := (H',\mathcal{U}',\{t,x\},\{(a',b),(a',a),(b,t_2)\}\cup B_1)$ is the required template. $\square$

Proof of Lemma 25. Let $\mathbb{T}$ be a 4-tuple $(H,\mathcal{U},\{a,b\},X)$ where $(H,\mathcal{U})$ is a U-graph, a,b are vertices of H, and $X\subseteq E(H)$. Let $\vec{H}$ be the LR-graph corresponding to $(H,\mathcal{U})$. We identify the following possible properties for such a 4-tuple $\mathbb{T}$.

(P1) $(H,\mathcal{U})$ arises from (G,Σ);

(P2) $\mathcal{B}_{\vec{H}}(X\cup\{L\})=\{a,b\}$;

(P3) $\{a,b\}\cap\mathcal{U}=\emptyset$;

Properties (P4), (P5), (P6), (P7) state that there does *not* exist a U-graph $(H,\mathcal{U}')$, where $\mathcal{U}'=(s_1',t_1',s_2',t_2')$, that is equivalent to $(H,\mathcal{U})$ and which has

(P4) edges $e=(s_1',a)$ and $f=(t_1',b)$ that form a cut of $\vec{H}$ separating L and R.

(P5) edges $e=(s_1',a)$ and $f=(b,t_2')$ that form a cut of $\vec{H}$ separating L and R.

(P6) edges $e=(s_1',a)$ and $f=(b,t_2')$ where $\deg_H(s_1')=\deg_H(t_2')=1$.

(P7) $\deg_H(s_1')=\deg_H(t_2')=1$.

Moreover,

(P8) $\{a,b\}$ is not skewed;

Claim 1. *There exists* 4*-tuple* $\mathbb{T}$ *satisfying (P1)-(P6).*

Proof. We may assume from Lemma 37 that there exists disjoint blocking pairs $\{s,t\}$ and $\{a,b\}$ of (G,Σ) that are not twins. Since $\{s,t\}$ is a blocking pair there exists a signature Σ' of (G,Σ) where $\Sigma'\subseteq\delta_G(s)\cup\delta_G(t)$. Let $(H,\mathcal{U})$ be obtained by unfolding (G,Σ') on s,t. Remark 23 implies that $\{a,b\}$ is an intercepting pair of $(H,\mathcal{U})$. Because of Remark 38 and Lemma 39 we may assume, after possibly redefining $(H,\mathcal{U})$, that $\{a,b\}$ is a good intercepting pair of $(H,\mathcal{U})$. Hence, $\{a,b\}$ is displayed in the U-graph $(H,\mathcal{U})$ by some set $X\subseteq E(H)$. Then $\mathbb{T}:=(H,\mathcal{U},\{a,b\},X)$ satisfies (P1)-(P3). For each of (P4), (P5) and (P6), edges e and f form a signature of (G,Σ). It follows in each cases that $\{s,t\}$ and $\{a,b\}$ are twins, a contradiction. $\diamond$

By Claim 1 there exists a 4-tuple $\mathbb{T}=(H,\mathcal{U},\{a,b\},X)$ satisfying (P1)-(P6). Suppose that $\mathbb{T}$ does not satisfy (P8), i.e. there exists edges (a',a) and (b,b') of H, for some vertices a',b' of H, such that $\{(a',a),(b,b')\}$ form a cut of H separating L and R, and $(a',a)\in X$, $(b,b')\notin X$. Consider the following 4-tuples:

$$\mathbb{T}_1 := \big(H,\mathcal{U},\{a,b'\},X\cup\{(b,b')\}\big) \qquad \mathbb{T}_2 := \big(H,\mathcal{U},\{a',b\},X\setminus\{(a,a')\}\big).$$

Claim 2. *For $i = 1, 2$, $\mathbb{T}_i$ satisfy properties (P1), (P2), (P5), (P6) and (P8).*

Proof. Let $i \in [2]$. Since the U-graph corresponding to the first two components of $\mathbb{T}$ and $\mathbb{T}_i$ is the same, (P1) holds for $\mathbb{T}_i$. Property (P2) is easy to verify for $\mathbb{T}_i$. We leave it as an exercise to check that (P8) holds for $\mathbb{T}_i$. Denote by (s_1, t_1, s_2, t_2) the elements of $\mathcal{U}$. Suppose (P5) does not hold for $\mathbb{T}_1$. (P3) implies that $b \notin \{s_1, t_1\}$. Thus $e = (a', a)$ and $a' \in \{s_1, t_1\}$. We may assume $a' = s_1$ as the case $a' = t_1$ is similar (interchange s and t in the proof). Then $f = (b', t_2)$. Since $\{e, f\}$ and $\{(a', a), (b, b')\}$ are cuts of H, so is $\{(a', a), (b', t_2)\} \triangle \{(a', a), (b, b')\} = \{(b, b'), (b', t_2)\}$. Hence, $\deg_H(b') = 2$ or b' is a cut vertex of H, a contradiction to Lemma 33. Hence, (P5) holds for $\mathbb{T}_1$. The proof to show that (P5) holds for $\mathbb{T}_2$ is similar, hence we omit it. Suppose (P6) does not hold for $\mathbb{T}_1$. (P3) implies that $b \notin \{s_1, t_1\}$. Thus $e = (a', a)$ and $a' \in \{s_1, t_1\}$. We may assume $a' = s_1$ as the case $a' = t_1$ is similar. Then $f = (b', t_2)$. Since $1 = \deg_H(s_1) = \deg_H(a')$ and since $\{(a', a), (b, b')\}$ is a cut of H separating a', b and a, b', Lemma 33 implies that $\deg_H(b) = 1$. But then $b = t_1$ a contradiction. Hence, (P6) holds for $\mathbb{T}_1$. The proof to show that (P6) holds for $\mathbb{T}_2$ is similar, hence we omit it. $\diamond$

Claim 3. *For some $i \in [2]$, $\mathbb{T}_i$ satisfies (P3) and (P4).*

Proof. If (P3) does not hold for $\mathbb{T}_1$ then $b' \in \{s_2, t_2\}$. If (P3) does not hold for $\mathbb{T}_2$ then $a' \in \{s_1, t_1\}$. Note, $\{a', b'\}$ is an intercepting pair since $\{(a, a'), (b, b')\}$ is a cut of $\vec{H}$ separating L and R. If $b' = s_2'$ and $a' = s_1'$ then $H - \{s_1, s_2\}$ has no $s_1 - s_2$ and no $t_1 - t_2$ paths, hence, the vertex s (corresponding to s_1, s_2) is a blocking vertex of (G, Σ), contradicting (h2). Similarly, $b' = t_2$ and $a' = t_1$ is not possible either. Thus, for $i \in [2]$, $a' = s_i$, $b' = t_{(3-i)}$, contradicting (P5). Because of Remark 40 and Lemma 41 we may assume that $(H, \mathcal{U})$ is not a Ladder or a Widget. If (P4) does not hold for $\mathbb{T}_1$ and (P4) does not hold for $\mathbb{T}_2$ then $(H, \mathcal{U})$ is a ladder, a contradiction. Thus we may assume, up to equivalence that (P3) does not hold for $\mathbb{T}_1$ and that (P4) does not hold for $\mathbb{T}_2$. But then $(H, \mathcal{U})$ is a Widget, a contradiction. $\diamond$

Claim 4. *There exists a 4-tuple $\mathbb{T}$ satisfying (P1)-(P8).*

Proof. By Claim 2 and Claim 3 for some $i \in [2]$, $\mathbb{T}_i$ satisfies all of (P1)-(P6) and (P8). Moreover, observe that if $\mathbb{T}$ satisfies (P7) then so does $\mathbb{T}_i$. It follows that it is sufficient to construct a 4-tuple $\mathbb{T}'$ that satisfies (P1)-(P7). Because of $\mathbb{T}_i$ we may assume that $\mathbb{T} = (H, \mathcal{U}, \{a, b\}, X)$ satisfies (P1)-(P6) and (P8). Let $(H', \mathcal{U}')$ be obtained from $(H, \mathcal{U})$ by shifting X. Recall, $\mathcal{U} = (s_1, t_1, s_2, t_2)$ and let s (resp. t) denote the vertices of G corresponding to s_1, s_2 (resp. t_1, t_2). Then let $\mathbb{T}' := (H', \mathcal{U}', \{s, t\}, \bar{X})$. Remark 35 implies that (P1) holds for $\mathbb{T}'$. It is easy to check that (P2) holds for $\mathbb{T}'$. Properties (P3), (P4), (P5), (P6), (P8) for $\mathbb{T}$ imply respectively properties (P3), (P4), (P6), (P5), (P7) for $\mathbb{T}'$. $\diamond$

Let $\mathbb{T} = (H, \mathcal{U}, \{a, b\}, X)$ be obtained from Claim 4. We claim that $\mathbb{T}$ is the required template. First $\{a, b\}$ is a good intercepting pair because of (P2) and (P3). Suppose (T2) does not hold. Then up to equivalence we either have, $\deg_H(s_1) = \deg_H(t_2) = 1$ or $\deg_H(s_1) = \deg_H(s_2) = 1$. In the former case we contradict (P7) in the latter case, s has degree 2 in G, contradicting (h1). Condition (T3) holds by (P8). Suppose, for a contradiction, that (T1) does not hold. Then we may assume, up to equivalence, that $H[X \setminus E_0]$ is not connected. Lemma 33 implies that $H[X \setminus E_0]$ consists of independent edges (s_1, a) and (t_1, b) (after possibly interchanging the labels of a and b), contradicting (P4). $\square$

4.4 Proof of Lemma 30

The goal of this section is to show that if a 1-extremal template arises from a relevant signed graph then that signed graph is an Octahedron. Before we prove this result we shall require some preliminaries.

We say that a template $\mathbb{T} = \big(H, \mathcal{U}, \{a, b\}, X\big)$, with $\mathcal{U} = (s_1, t_1, s_2, t_2)$, is a *flower template* (see Figure 9) if there exists vertices $c, d \notin \mathcal{U} \cup \{a, b\}$, a partition A, B of X, and a partition C, D of $\bar{X}$ such that the following conditions hold.

(F1) $s_1 \in \mathcal{I}(A), t_1 \in \mathcal{I}(B), s_2 \in \mathcal{I}(C), t_2 \in \mathcal{I}(D)$;

(F2) $\mathcal{B}(A) = \{a, c\}, \mathcal{B}(B) = \{b, c\}, \mathcal{B}(C) = \{b, d\}, \mathcal{B}(D) = \{a, d\}$;

(F3) $H[A], H[B], H[C], H[D]$ are connected.

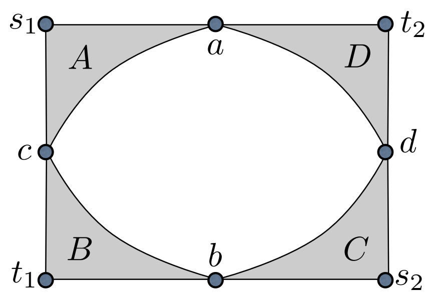


Figure 9: Flower template

Lemma 42. *Let $\mathbb{T} = \big(H, \mathcal{U}, \{a, b\}, X\big)$ be a template, with $\mathcal{U} = (s_1, t_1, s_2, t_2)$, that arises from a relevant signed graph (G, Σ). Suppose $\mathbb{T}$ has a bad intercepting pair $\{c, d\}$ of Type A. Then, up to equivalence, $\mathbb{T}$ is a flower template.*

Proof.

Claim 1. *We may assume (up to equivalence), that there exists $Z \subseteq E(H)$ where:*

(1) $s_1, t_2 \in \mathcal{I}(Z)$, $s_2, t_1 \in \mathcal{I}(\bar{Z})$,

(2) $\mathcal{B}(Z) = \{c, d\}$,

(3) there exists an $s_1 - t_2$ path P in $H[Z] - \{c, d\}$.

Proof. Since $\{c, d\}$ is a bad intercepting pair, Lemma 34 implies that there exists an $s_i - t_{(3-i)}$ path P in $H - \{c, d\}$, for some $i \in [2]$. We may assume (up to equivalence) that $i = 1$. Let Z_0 be the edges in the component of $H - \{c, d\}$ that contains s_1, t_2. Define, $Z := Z_0 \cup \{(v, c) : v \in V(Z_0)\} \cup \{(v, d) : v \in V(Z_0)\}$. Since $\{c, d\}$ is an intercepting pair $s_2, t_1 \notin V(Z)$. By construction $\mathcal{B}(Z) \subseteq \{c, d\}$. Lemma 33 implies that $\mathcal{B}(Z) = \{c, d\}$. $\diamond$

Claim 2. *After possibly exchanging the role of a and b we have $a \in \mathcal{I}(Z), b \in \mathcal{I}(\bar{Z})$.*

Proof. We claim that there exists a $t_1 - s_2$ path Q of $H[\bar{Z}]$. For otherwise $H[\bar{Z}]$ has edge sets Z_1, Z_2, corresponding to components of $H[\bar{Z}]$ with $t_1 \in V(Z_1)$ and $s_2 \in V(Z_2)$. Since H is connected (Lemma 33), we may assume that $c \in V(Z_1)$ and $d \in V(Z_2)$. Lemma 33 then implies that $|Z_1| = |Z_2| = 1$, contradicting property (T2) of templates. Thus we have vertex disjoint $\{s_1, t_1\} - \{s_2, t_2\}$ paths P, Q of H (where P is given in Claim 1). As $\{a, b\}$ is a good intercepting pair we may assume that $a \in V(P)$ and $b \in V(Q)$. ◇

Claim 3. *After possibly exchanging the role of c and d we have $c \in \mathcal{I}(X), d \in \mathcal{I}(\bar{X})$.*

Proof. Otherwise we may assume (up to equivalence) that $c, d \in \mathcal{I}(\bar{X})$. Because of Claim 1 parts (1),(2), every $s_1 - t_1$ path intersects one of $\{c, d\}$. In particular, there exists no $s_1 - t_1$ path in $H[X]$. Hence, $H[X]$ is not connected, contradicting property (T1). ◇

Now define,

$$A := X \cap Z \qquad B := X \cap \bar{Z} \qquad C := \bar{X} \cap \bar{Z} \qquad D := \bar{X} \cap Z.$$

Claim 1, Claim 2 and Claim 3 imply that properties (F1) and (F2) of flower templates hold. Suppose (F3) does not hold. Up to symmetry it suffices to consider the case where A can be partitioned into A_1, A_2 where $V(A_1) \cap V(A_2) = \emptyset$. Then we may assume that $a \in V(A_1)$ and $c \in V(A_2)$. Then $s_1 \notin V(A_1)$ or $s_1 \notin V(A_2)$. In the former case A_1 contradicts Lemma 33, in the latter case A_2 does. □

Proof of Lemma 30. Suppose $\mathbb{T}$ is of the form $(H, \mathcal{U}, \{a, b\}, X)$, with $\mathcal{U} = (s_1, t_1, s_2, t_2)$. Since $\mathbb{T}$ is of Type 1, there is a bad intercepting set $W = \{c, d\}$ of $(H, \mathcal{U})$. By Lemma 42, $\mathbb{T}$ is a flower template. It remains to show that A, B, C, D are solid separations of G.

Claim. *Each of A, B, C, D is either solid or a triad.*

Proof. As we can consider equivalent templates, it suffices to show the result for A. Because of Lemma 36 we only need to verify the following statements,

(1) there exists an $s_1 - c$ path of $H[A] - a$,

(2) there exists an $s_1 - a$ path of $H[A] - c$,

(3) there exists an $a - c$ path of $H[A] - s_1$.

Suppose (1) does not hold. Then Lemma 33 implies that A consists of edges (s_1, a) and (c, a). Observe that $H - \{a, d\}$ has no $s_1 - s_2$ path and no $t_1 - t_2$ path. Hence, $\{a, d\}$ is an intercepting pair. By (T2), $\deg_H(s_2) \geq 2$. It follows, by Lemma 33, that there exists a $b - s_2$ path of $H[C] - d$. Since $H[B]$ is connected, there exists a $t_1 - s_2$ path of $H - \{a, d\}$. Hence, $\{a, d\}$ is a bad intercepting pair. In particular, $\{a, d\}$ is a bad intercepting set of Type C, contradicting the fact that $\mathbb{T}$ is a 1-extremal.

Suppose (2) does not hold. Consider first the case where s_1 and t_1 do not both have degree 1 in H. By Remark 35, $(H, \mathcal{U}')$ with $\mathcal{U}' = (s_1, t_2, s_2, t_1)$ also arises from (G, Σ). Observe, now that $\mathbb{T}' :=$

$(H,\mathcal{U}',\{c,d\},A\cup D)$ is a Flower template. Moreover, paths s_1-a of $H[A]-c$ play the same role in the template $\mathbb{T}'$ as paths s_1-c of $H[A]-a$ in the template $\mathbb{T}$. Hence, following the argument in the previous case, we deduce again that $\mathbb{T}$ is not 1-extremal, a contradiction. If both s_1 and t_1 have degree 1 in H, we first construct a new U-graph $(\hat{H},\hat{\mathcal{U}})$ from $(H,\mathcal{U})$ by shifting the unique edge of H with end in t_1. Then we proceed with $(\hat{H},\hat{\mathcal{U}})$ as previously. (The shift operation is required for otherwise $\mathbb{T}'$ is not a template as it will violate condition (T2).)

Suppose (3) does not hold. Then $H-\{s_1,s_2,d\}$ has no s_1-s_2 and t_1-t_2 path. Hence, $\{s_1,s_2,d\}$ is an intercepting set. Moreover, there exists a t_1-s_2 path in $H-\{s_1,d\}$ and a t_1-t_2 path in $H-\{s_2,d\}$. This implies that $\{s_1,s_2,d\}$ is a bad intercepting set of Type B, contradicting the fact that $\mathbb{T}$ is 1-extremal. $\diamond$

It follows from the Claim that in order to complete the proof we need to show that none of A,B,C,D are triads. As we can consider equivalent templates, it suffices to show the result for A. Suppose for a contradiction that A is a triad $(z,s_1),(z,a),(z,c)$ of H. Note that B is not a triad for otherwise $\deg_G(c)=2$. Thus B is solid in G. It follows that $\mathbb{T}':=(H,\mathcal{U},\{z,b\})$ is a template. Observe that $\{z,d\}$ is an intercepting pair. Because B is solid and C is either a triad or solid, there exists a t_1-s_2 path of $H-\{z,d\}$. It follows that $\{z,d\}$ is a bad intercepting pair. Then $\{z,d\}$ is a bad intercepting pair of Type C for $\mathbb{T}'$. We deduce that $\mathbb{T}$ is not 1-extremal, a contradiction. $\square$

4.5 Proof of Lemma 32

The goal of this section is to show that if a 5-extremal template arises from a relevant signed graph then that signed graph is a Kite. Before we prove this result we shall require some preliminaries.

We say that a template $\mathbb{T}=\big(H,\mathcal{U},\{a,b\},X\big)$, with $\mathcal{U}=(s_1,t_1,s_2,t_2)$, is a *strip template* (see Figure 10) if there exist vertex $c\notin U\cup\{a,b\}$, an edge $\Omega=(t_1,a)$, a partition $A,\{\Omega\}$ of X and a partition B,C of $\bar{X}$ such that the following conditions hold.

(S1) $s_1\in\mathcal{I}(A), s_2\in\mathcal{I}(C), t_2\in\mathcal{I}(B)$,

(S2) $\mathcal{B}(A)=\{a,b\},\{b,c\}\subseteq\mathcal{B}(B)\subseteq\{a,b,c\},\mathcal{B}(C)=\{a,c\}$.

Note that, we get two types of strip templates depending on whether $\mathcal{B}_H(B)=\{b,c\}$ or $\mathcal{B}_H(B)=\{a,b,c\}$.

Lemma 43. *Let* $\mathbb{T}=\big(H,\mathcal{U},\{a,b\},X\big)$ *be a template, with* $\mathcal{U}=(s_1,t_1,s_2,t_2)$*, that arises from a relevant signed graph* (G,Σ)*. Suppose* $\mathbb{T}$ *has a bad intercepting pair* $\{a,c\}$ *of Type C. Then, up to equivalence,* $\mathbb{T}$ *is a strip template.*

Proof. We may assume, up to equivalence, that $c\in\mathcal{I}_H(\bar{X})$. Since $\{a,c\}$ is a bad intercepting pair, Lemma 34 implies that there exists an $s_i-t_{(3-i)}$ path in $H-\{a,c\}$ for some $i\in[2]$. Up to equivalence, we may assume that there exists an s_1-t_2 path in $H-\{a,c\}$. In particular there exists an s_1-b path P of $H[X]-a$ and a $b-t_2$ path Q of $H[\bar{X}]-\{a,c\}$.

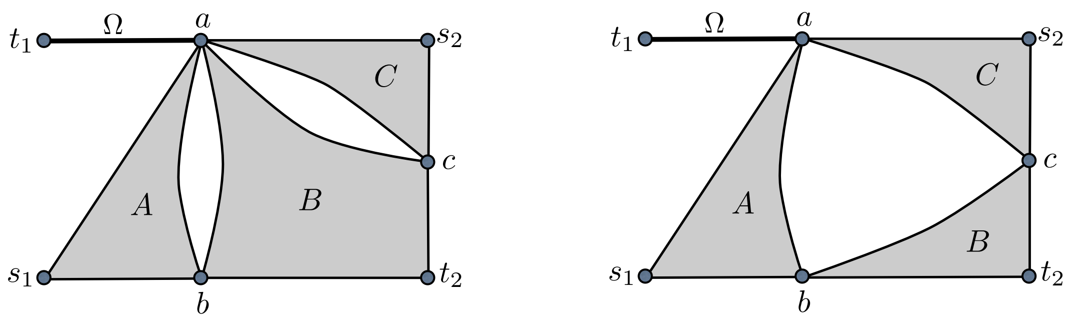


Figure 10: Two types of strip template

Because of Q and the fact that $\{a, c\}$ is an intercepting pair, there is no $t_1 - b$ path of $H[X] - a$. Thus there exists a partition Ω, A of X such that $P \subseteq A$, $t_1 \in \mathcal{I}(\Omega)$, and $V(\Omega) \cap V(A) \subseteq \{a\}$, in particular $b \notin V(\Omega)$. By Lemma 33, Ω consists of a single edge (t_1, a). We have $\{b\} \subseteq \mathcal{B}(A) \subseteq \{a, b\}$. If $\mathcal{B}(A) = \{b\}$, Lemma 33 implies that A consists of a single edge. But then $H[X] - E_0$ is not connected, contradicting (T1).

Because of P, Q and the fact that $\{a, c\}$ is an intercepting pair, there is no $\{b, t_2\} - s_2$ path of $H[\bar{X}] - \{a, c\}$. Hence, there exists a partition B, C of $\bar{X}$ such that $Q \subseteq B$, $s_2 \in \mathcal{I}(C)$, and $V(B) \cap V(C) \subseteq \{a, c\}$. If $\mathcal{B}(C) \subset \{a, c\}$ then Lemma 33 implies that $\deg(s_2) = 1$, contradicting (T2). Path Q implies that $b \in \mathcal{B}(B)$. Clearly, $\mathcal{B}(B) \subseteq \{a, b, c\}$. Moreover, as $c \in \mathcal{B}(C)$, $c \in \mathcal{B}(B)$. □

Let $(H, \mathcal{U})$ be a U-graph and let $X \subseteq E(G)$. We say that a path P is an $s - t|_{\mathcal{U}}\ X$ (or is of type $s - t|_{\mathcal{U}}\ X$) if P is an $s - t$ path of $H[X]$ avoiding all vertices in $(\mathcal{U} \cup \mathcal{B}(X)) \setminus \{s, t\}$.

Lemma 44. *Let $\mathbb{T} = (H, \mathcal{U}, \{a, b\}, X)$ be a template, with $\mathcal{U} = (s_1, t_1, s_2, t_2)$, that arises from a relevant signed graph (G, Σ). Suppose $\mathbb{T}$ is a strip template where $\{a, c\}$ is a bad intercepting pair. Then there exist paths $P_1, \ldots, P_6$ of H of the following types.*

$$P_1 : s_1 - a|_{\mathcal{U}}\ A \qquad P_2 : s_1 - b|_{\mathcal{U}}\ A \qquad P_3 : b - c|_{\mathcal{U}}\ B$$
$$P_4 : t_2 - b|_{\mathcal{U}}\ B \qquad P_5 : s_2 - a|_{\mathcal{U}}\ C \qquad P_6 : s_2 - c|_{\mathcal{U}}\ C$$

Moreover, either

(1) there exists a path of type $a - t_2|_{\mathcal{U}}\ B$, or

(2) there exists a path of type $a - b|_{\mathcal{U}}\ A \cup B$ and a path of type $a - c|_{\mathcal{U}}\ B \cup C$.

Proof. Since $\mathbb{T}$ satisfies property (T1), paths P_1 and P_2 exist. Since $\{a, c\}$ is a bad intercepting pair, by Lemma 34(1) and the structure of strip templates, there exists an $s_1 - t_2$ path P of $H - \{a, c\}$. Then let $P_4 := P[b, t_2]$. Since $\mathbb{T}$ satisfies property (T2), $\deg_H(s_2) \geq 2$. The existence of paths P_5 and P_6 then follows from Lemma 33.

Claim. *The following are not intercepting sets: $\{a, t_1, t_2\}$, $\{b, s_1, s_2\}$ and $\{c, s_1, s_2\}$.*

Proof. Observe that $\{a, s_1, s_2\}$ is an intercepting set of H. Hence, $\{a, s\}$ is a blocking pair of (G, Σ). Since $\{a, b\}, \{s, t\}, \{a, c\}$ are also blocking pairs, and since by choice (h4), of (G, Σ) has no blocking triple, $\{a, t\}, \{b, s\}$ and $\{c, s\}$ are not blocking pairs. $\diamond$

Since $\{a, t_1, t_2\}$ is not an intercepting triple, there exists an $s_1 - s_2$ path P of $H - \{a, t_1, t_2\}$. Then let $P_3 := P[b, c]$. Hence, the first part of the statement holds.

It remains to show that either outcome (1) or (2) holds. Since $\{s_1, s_2, c\}$ is not an intercepting pair, there exists a $t_1 - t_2$ paths of $H - \{s_1, s_2, c\}$. In particular, there exists an $a - t_2$ path Q in $H[A \cup B] - \{s_1, c\}$. If $b \notin V(Q)$ then outcome (1) holds, otherwise we let $Q' := Q[a, b]$. Since $\{s_1, s_2, b\}$ is not an intercepting pair, there exists a $t_1 - t_2$ path of $H - \{s_1, s_2, b\}$. In particular, there exists an $a - t_2$ path R in $H[B \cup C] - \{s_2, b\}$. If $c \notin V(R)$ then outcome (1) holds, otherwise we let $R' := R[a, c]$. Now, Q' and R' imply that outcome (2) holds. $\square$

Proof of Lemma 32. Suppose $\mathbb{T}$ is of the form $(H, \mathcal{U}, \{a, b\}, X)$, with $\mathcal{U} = (s_1, t_1, s_2, t_2)$. Since $\mathbb{T}$ is of Type 5, there is a bad intercepting set $W = \{a, c\}$ of $(H, \mathcal{U})$. By Lemma 43, $\mathbb{T}$ is a strip template. Let $P_1, \ldots, P_6$ be the paths given by Lemma 44. Since there is no internal pivot, there exists an $s_2 - t_2$ path P in $H[B \cup C] - \{a, b\}$. Define $P_7 := P[t_2, c]$.

Suppose outcome (1) of Lemma 44 occurs. Then let P_8 denote the $a - t_2|_{\mathcal{U}}$ B path of H. It follows that (G, Σ), obtained by folding $(H, \mathcal{U})$, is a Kite of Type II, with $P_1, \ldots, P_8$ satisfying the required conditions. (See Section 1.3.4.) Thus we may assume outcome (1) does not occurs. Thus outcome (2) occurs, i.e. there exists an $a - b|_{\mathcal{U}}$ $A \cup B$ path P_8 and an $a - c|_{\mathcal{U}}$ $B \cup C$ path P_9 of H.

Consider first the case where $a \notin \mathcal{B}_H(B)$. Then P_8 is an $a - b|_{\mathcal{U}}$ A path of H and P_9 is an $a - c|_{\mathcal{U}}$ C path of H. Paths P_1, P_2, P_8 and Lemma 36 implies that A is solid or a triad in G. Paths P_5, P_6, P_9 and Lemma 36 implies that C is solid or a triad of G. Paths P_3, P_4, P_7 and Lemma 36 implies that B is solid or a triad of G. However, (T2) implies that B and C are not a triad of G, hence they are solid. It follows that (G, Σ), obtained by folding $(H, \mathcal{U})$, is a Kite of Type I. (See Section 1.3.3.)

Consider now the case where $a \in \mathcal{B}_H(B)$. Since, there is no $a - t_2|B$ path of H, we can partition B into B' and D such that $\mathcal{B}_H(B') = \{b, c\}$, $t_2 \in \mathcal{I}_H(B')$ and $\mathcal{B}_H(D) \subseteq \{a, b, c\}$. We may assume that $\mathcal{I}_G(D) \neq \emptyset$ for otherwise, (h1) implies that D consists of edges with both endpoints in $\{a, b, c\}$. But then we can redefine A (resp. B, C) so as to contain edges with ends a, b (resp. b, c and a, c), in which case $D = \emptyset$ and we are in the previous case. Let $z \in \mathcal{I}_G(D)$. By (h1) there exists three $z - \{a, b, c\}$ paths, that only share vertex z, included in D. It follows from Lemma 36 that D is a triad or solid. Hence, (G, Σ), obtained by folding $(H, \mathcal{U})$, is a Kite of Type III, with $P_1, P_2, P_4, P_5, P_6, P_7$ satisfying the required conditions. (See Section 1.3.5.) $\square$

4.6 Proof of Lemma 31

The goal of this section is to show that if a 3-extremal template arises from a relevant signed graph then that signed graph is a Saucer. Before we prove this result we shall require some preliminaries.

We say that a template $\mathbb{T} = \big(H, \mathcal{U}, \{a,b\}, X\big)$, with $\mathcal{U} = (s_1, t_1, s_2, t_2)$, is a *swivel template* (see Figure 11) if there exist vertices $c, d \notin U \cup \{a,b\}$, an edge $\Omega = (t_1, a)$, a partition $A_1, A_2, \{\Omega\}$ of X and a partition C_1, C_2, D of $\bar{X}$ such that the following conditions hold.

(W1) $s_1 \in \mathcal{I}(A_1)$, $s_2 \in \mathcal{I}(C_1)$, $t_2 \in \mathcal{I}(D)$.

(W2) $\mathcal{B}(A_1) = \{a,d\}, \mathcal{B}(A_2) = \{a,d,b\}, \mathcal{B}(C_1) = \{a,c\}, \mathcal{B}(C_2) = \{a,c,b\}, \mathcal{B}(D) = \{a,b\}$.

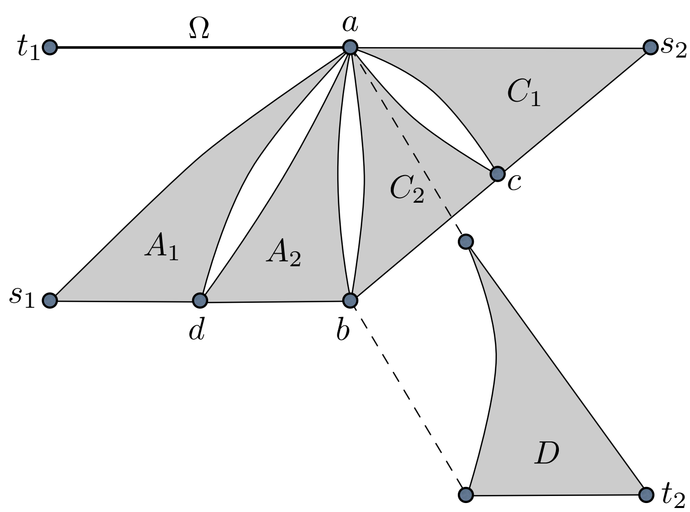


Figure 11: Swivel template

Lemma 45. *Let $\mathbb{T} = \big(H, \mathcal{U}, \{a,b\}, X\big)$ be a template, with $\mathcal{U} = (s_1, t_1, s_2, t_2)$, that arises from a relevant signed graph (G, Σ). Suppose $\mathbb{T}$ is of Type 3, then it is a swivel template. Moreover, there exists paths $Q_1, \ldots, Q_{10}$ of H of the following types.*

$Q_1 : s_1 - a\vert_{\mathcal{U}}\ A_1$	$Q_2 : s_1 - d\vert_{\mathcal{U}}\ A_1$	$Q_3 : d - b\vert_{\mathcal{U}}\ A_2$	$Q_4 : a - d\vert_{\mathcal{U}}\ A_1 \cup A_2$
$Q_5 : s_2 - a\vert_{\mathcal{U}}\ C_1$	$Q_6 : s_2 - c\vert_{\mathcal{U}}\ C_1$	$Q_7 : c - b\vert_{\mathcal{U}}\ C_2$	$Q_8 : a - c\vert_{\mathcal{U}}\ C_1 \cup C_2$
$Q_9 : t_2 - a\vert_{\mathcal{U}}\ D$	$Q_{10} : t_2 - b\vert_{\mathcal{U}}\ D$		

Proof of Lemma 31. Directly from Lemma 45 and the definition of folding. $\square$

We require a number of preliminaries before we can prove Lemma 45.

Lemma 46. *Let $\mathbb{T} = (H, \mathcal{U}, \{a,b\}, X)$ be a template, with $\mathcal{U} = (s_1, t_1, s_2, t_2)$, that is obtained by unfolding a relevant signed graph (G, Σ) on vertices s, t. Then $\mathbb{T}$ does not have both an internal pivot and an external pivot.*

Proof. Suppose that for a contradiction that $\mathbb{T}$ has both an internal pivot and an external pivot.

Claim 1. $\mathcal{I}(X) = \{s_1, t_1\}$ *and* $\mathcal{I}(\bar{X}) = \{s_2, t_2\}$.

Proof. It suffices to prove the statement for X as the proof for $\bar{X}$ is similar. Suppose for a contradiction there exists $z \in \mathcal{I}(X) \setminus \{s_1, t_1\}$. Lemma 33 implies that there exist $z - \{s_1, t_1, a, b\}$ minimal paths P_1, P_2, P_3 where $V(P_i) \cap V(P_j) = \{z\}$ for all $i, j \in [3]$, $i \neq j$. Thus for some $i, j \in [2]$ either: P_i is a $z - a$ path and P_j is a $z - b$ path, or P_i is a $z - s_1$ path and P_j is a $z - s_2$ path. Let $Q := P_i \cup P_j$. Then Q shows in the former (resp. latter) case that $\mathbb{T}$ is not an external (resp. internal) pivot. $\diamond$

Properties (T1) and (T2) of templates imply that at most one vertex of U has degree 1 in H. Suppose no vertex of U has degree 1. Then $E(H) = \{(s_i, a), (t_i, a), (s_i, b), (t_i, b) : i = 1, 2\}$. In that case (G, Σ) has exactly two blocking pairs namely $\{a, b\}$ and $\{s, t\}$ which are displayed by H, contradicting (h5). Suppose exactly one vertex, say s_1, of U has degree 1. Then $E(H) = \{(s_1, a), (t_1, a), (t_1, b), (s_2, a), (s_2, b), (t_2, a), (t_2, b)\}$. In that case (G, Σ) has exactly three blocking pairs namely, $\{a, b\}$, $\{s, t\}$ and $\{a, t\}$ which are displayed by H, contradicting (h5). $\square$

Lemma 47. *A flower template $\mathbb{T} = (H, \mathcal{U}, \{a, b\}, X)$, with $\mathcal{U} = (s_1, t_1, s_2, t_2)$, has no internal pivot.*

Proof. Suppose for a contradiction that $\mathbb{T}$ has an internal pivot. Then there is no $s_1 - t_1$ path of $H[X] - \{a, b\}$. Thus either there is no $s_1 - c$ path of $H[A] - a$ or there is no $t_1 - c$ path of $H[B] - b$. Lemma 33 implies that $\deg_H(s_1) = 1$ in the former case and $\deg_H(t_1) = 1$ in the latter one. Similarly, $\deg_H(s_2) = 1$ or $\deg_H(t_2) = 1$. But this contradicts property (T2) of templates. $\square$

Proof of Lemma 45. Lemma 43 implies that $\mathbb{T}$ is a strip template. Denote by $P_1, \ldots, P_6$ the paths in Lemma 44. Let D_0 be the edge set of the component of $H[\bar{X}] - \{a, b\}$ that contains t_2. Define $D := D_0 \cup \{(a, v) : v \in V(D_0)\} \cup \{(b, v) : v \in V(D_0)\}$. Since we have an internal pivot, $s_2 \notin V(D)$. Because of P_6, c is also not in $V(D)$. Hence, $D \subseteq B$. Define C' as $\bar{X} \setminus D$. Note that, $C' \supseteq C$ and because of path P_6, $c \in V(C')$. Let F be edge set of the component of $H[C'] - \{a, c\}$ that contains s_2. Let $C_1 := F \cup \{(a, v) : v \in V(F)\} \cup \{(c, v) : v \in V(F)\}$ and let $C_2 := C' \setminus C_1$. Then $V(C_1) \cap V(C_2) = \{a, c\}$. Let $Q_5 := P_5$, $Q_6 := P_6$, $Q_7 := P_3$, $Q_{10} = P_4$. Property (T2) implies that $\deg_H(t_2) > 1$. Together with Lemma 33 this implies that there exists a $t_2 - a$ path Q_9 in $H[D] - b$. Lemma 33 implies that $\deg_H(c) > 2$. Hence, there exists an $a - c$ path Q_8 in $H[C_1 \cup C_2] - \{s_2, b\}$.

Consider first the case where $\deg_H(s_1) > 1$. Remark 35 implies that the U-graph $(H, \mathcal{U}')$, with $\mathcal{U}' = (s_2, t_1, s_1, t_2)$ also arises from (G, Σ). Since $\mathbb{T}$ has an internal pivot, $\{a, b\}$ is a good intercepting pair of $(H, \mathcal{U}')$. It follows that $\mathbb{T}' := (H, \mathcal{U}', \{a, b\}), X')$ is a template where $X' = C_1 \cup C_2 \cup \{\Omega\}$. Lemma 24 implies that there exists a bad intercepting set W of $(H, \mathcal{U}')$. W is not of Type A, for otherwise Lemma 42 implies that $\mathbb{T}$ is a flower template, contradicting Lemma 47. Lemma 46 implies that $\mathbb{T}$ has no external pivot. Because $\mathbb{T}$ is 3-extremal, W is not of an intercepting pair of Type B of $\mathbb{T}'$. Thus, Lemma 27 implies that W is of Type C for some template equivalent to $\mathbb{T}'$. Note, that A plays the same role in $\mathbb{T}'$ as C in $\mathbb{T}$. Hence $W = \{a, d\}$ for some vertex $d \in V(A)$. By the same argument as above, we deduce that there exist a partition A_1, A_2 of A and a vertex d such that $s_2' \in \mathcal{I}(A_1)$, $a, b \in V(A_2)$ and $V(A_1) \cap V(A_2) = \{a, d\}$. Then Q_1, Q_2, Q_3, Q_4 play the say role for $A_1 \cup A_2$ as paths Q_5, Q_6, Q_7, Q_8 for $C_1 \cup C_2$. If $\deg_H(t_1) = 1$ have degree 1, we first construct a new U-graph $(\hat{H}, \hat{\mathcal{U}})$ from $(H, \mathcal{U})$ by shifting the unique edge of H with end in s_1. Then we proceed with $(\hat{H},\hat{\mathcal{U}})$ as previously. (The shift operation is required for otherwise $\mathbb{T}'$ is not a template as it will violate condition (T2).) $\square$

5 Proof of Proposition 13

The goal of this section is to show that nearly 3-connected Octahedrons, Kites, Saucers, and Pinwheels are timid.

5.1 The case of Pinwheels

Lemma 48. *Pinwheels are timid.*

Proof. Let (G,Σ) be a Pinwheel with hub h and let $H := G - h$. Then H is a maximal flower $\mathbb{F} = (B_1,\dots,B_r,u_1,\dots,u_r)$. Suppose for a contradiction that (G,Σ) is not timid, i.e. for some signature Σ' and vertices s,t we can obtain an LR-graph $\vec{H}$ by unfolding (G,Σ') on s,t such that every every blocking pair can be displayed as an LR-separator. Remark 3 states that the blocking pairs of (G,Σ) are the sets $\big\{\{h,u_i\} : i \in [r]\big\}$. Hence, may assume, up to relabeling the petals and attachments, that $s = h$ and $t = u_1$. Moreover, we can assume that $\Sigma' \cap E(H) = \delta_H(u_1) \cap B_1$. Let $\hat{\imath}$ is the largest index $i \in [r]$ for which there exists an edge $(h,w) \in \Sigma'$ where $w \in V_H(B_i) \setminus \{u_i,u_1\}$. Let $\hat{\jmath}$ is the smallest index $j \in [r]$ for which there exists an edge $(h,w) \in E(G) \setminus \Sigma'$ where $w \in V_H(B_j) \setminus \{u_{j+1},u_1\}$. Consider first the case where there is no edge $(h,u_1) \in \Sigma'$ in G. Since (G,Σ) is not 1-degenerate (see Section 1.3.7) we have that $\hat{\imath} > \hat{\jmath}$. It can be readily checked now that the blocking pair $\{h,u_{i+1}\}$ is not the boundary of any LR-separation of $\vec{H}$, a contradiction. Consider now the case where there is an edge $e = (h,u_1) \in \Sigma'$ in G. Since (G,Σ) is not 1-degenerate, $\hat{\imath} > 1$ and $\hat{\jmath} < r$. By the definition of unfolding e has either endpoints corresponding to the head of L and to the tail of R, or endpoints corresponding to the tail of L and the head of R. We consider the former case only, as the proof for the other case is similar. It can be readily checked now (as $\hat{\imath} > 1$) that $\{h,u_2\}$ is not the boundary of any LR-separation of $\vec{H}$, a contradiction. $\square$

5.2 Outline of the proof for the remaining cases

It remains to prove that nearly 3-connected Octahedrons, Kites, and Saucers are timid. The next proposition states that if we can display all blocking pairs as LR-separators, then it can "essentially" be done by unfolding on an arbitrary blocking pair.

Lemma 49. *Let (G,Σ) be a nearly 3-connected signed graph with no blocking vertex. Suppose (G,Σ) is not timid and let $\{s,t\}$ be a blocking pair. Then there exists a signature Σ' of (G,Σ) such that the U-graph $(H,\mathcal{U})$, where $\mathcal{U} = (s_1,t_1,s_2,t_2)$, obtained by unfolding (G,Σ') on s,t satisfies one of the following,*

(1) all blocking pairs of (G,Σ) are displayed by $(H,\mathcal{U})$, or

(2) up to equivalence, $deg_H(s_1) = 1$ and all blocking pairs of (G, Σ) are displayed in the U-graph obtained form $(H, \mathcal{U})$ by shifting the unique edge of H incident to s_1.

Let (G, Σ) be a signed graph and let $\{s, t\}$ be a blocking pair. We say that $\{s, t\}$ is *special* if there exists a non-trivial separation X of G where $\mathcal{B}(X) = \{s, t\}$. The next proposition indicates that there are, up to equivalence, at most two U-graphs that can be obtained by unfolding on a non-special blocking pair.

Lemma 50. *Let (G, Σ) be a nearly 3-connected signed graph with no blocking vertex and with a blocking pair $\{s, t\}$ that is not special. Let $\big(H, (s_1, t_1, s_2, t_2)\big)$ be the U-graph obtained by unfolding (G, Σ) on s, t. Let Σ' be a signature of (G, Σ) where $\Sigma' \subseteq \delta(s) \cup \delta(t)$. Then the U-graph obtained by unfolding (G, Σ') on s, t is equivalent to one of $\big(H, (s_1, t_1, s_2, t_2)\big)$ or $\big(H, (s_2, t_1, s_1, t_2)\big)$.*

We postpone the proof of these results until the next subsection. Using these last two lemmas we are now ready for the main result in this section,

Proof of Proposition 13. Let (G, Σ) be a nearly 3-connected signed graph that is either an Octahedron, a Kite, or a Saucer. Because of Lemma 48 is suffices to show that (G, Σ) is timid. If (G, Σ) is an Octahedron (resp. Kite of Type I, a Kite of Type II, a Kite of Type III or a Saucer) let s, t and Σ be as defined in Section 1.3.2 (resp. 1.3.3, 1.3.4, 1.3.5, 1.3.6). Let $\big(H, (s_1, t_1, s_2, t_2)\big)$ be the U-graph obtained from (G, Σ) by unfolding on s, t. Then $\mathbb{T} = \big(H, (s_1, t_1, s_2, t_2), \{a, b\}, X\big)$ is a template where $X = A \cup B$ if (G, Σ) is an Octahedron, $X = A \cup \{\Omega\}$ if (G, Σ) is a Kite, and $X = A_1 \cup A_2 \cup \{\Omega\}$ if (G, Σ) is a Saucer. If (G, Σ) is an Octahedron then $\mathbb{T}$ is a flower template (see Section 4.4). If (G, Σ) is a Kite then $\mathbb{T}$ is a strip template (see Section 4.5). If (G, Σ) is a Saucer then $\mathbb{T}$ is a swivel template (see Section 4.6).

Claim 1. $\big(H, (s_1, t_1, s_2, t_2)\big)$ *does not display all blocking pairs.*

Proof. Suppose (G, Σ) is an Octahedron. Since A and D are solid, there exists an $s_1 - t_2$ path of $H - \{c, d\}$. It follows that the intercepting pair $\{c, d\}$ is bad. Suppose (G, Σ) is a Boat, a Kite or a Saucer. We will show in each case that there exists an $s_1 - t_2$ path P of $H - \{a, c\}$ which implies that the intercepting pair $\{a, c\}$ is bad. For a Kite of Type I, A is a solid separation, or a triad of G, and B is a solid separation of G. Hence, there exists an $s_1 - t_2$ path P in $H[A \cup B] - \{a, c\}$ as required. For a Kite of Type II or Type III let $P := P_2 \cup P_4$. For a Saucer, let $P := Q_2 \cup Q_3 \cup Q_{10}$. $\diamond$

Claim 2. $\big(H, (s_2, t_1, s_1, t_2)\big)$ *does not display all blocking pairs.*

Proof. If (G, Σ) is a Saucer, then $\big(H, (s_1, t_1, s_2, t_2)\big)$ and $\big(H, (s_2, t_1, s_1, t_2)\big)$ are the same up to interchanging A_1 and A_2 with C_1 and C_2, respectively. Then result then follows from Claim 1. Suppose (G, Σ) is an Octahedron, a Boat, or a Kite. We will show in each case that there exists an $s_2 - t_2$ path P of $H - \{a, b\}$. Since P has endpoints that are the first and fourth element of (s_2, t_1, s_1, t_2), this will imply that $\{a, b\}$ is not displayed in $\big(H, (s_2, t_1, s_1, t_2)\big)$. For an Octahedron the existence of P follows from the fact that C and D are solid. For a Kite of Type I, the existence of P follows from the fact that B and C are solid. For a Kite of Type II or of Type III, let $P := P_6 \cup P_7$. $\diamond$

Suppose for a contradiction that (G, Σ) is not timid. Then by Lemma 49, for some signature Σ' of (G, Σ) the U-graph $(H', \mathcal{U}')$, where $\mathcal{U} = (s_1, t_1, s_2, t_2)$ obtained by unfolding (G, Σ') on s, t satisfies one of the following,

(1) all blocking pairs of (G, Σ) are displayed by $(H', \mathcal{U}')$, or

(2) up to equivalence, $deg_H(s_1) = 1$ and all blocking pairs of (G, Σ) are displayed in the U-graph obtained form $(H', \mathcal{U}')$ by shifting the unique edge of H' incident to s_1.

Suppose (1) occurs. It can be readily checked in each case (Octahedron, Kite, and Saucer) that $\{s, t\}$ is not special. It follows from Lemma 50 that $(H', \mathcal{U}')$ is equivalent to one of the U-graphs $\big(H, (s_1, t_1, s_2, t_2)\big)$ or $\big(H, (s_2, t_1, s_1, t_2)\big)$. Because of Claim 1 and Claim 2 neither of these U-graphs display all blocking pairs. It follows that $(H', \mathcal{U}')$ does not display all blocking pairs either, a contradiction. Suppose case (2) occurs. We showed $(H', \mathcal{U}')$ does not display all blocking pairs. It can be readily checked now that any U-graph obtained from $(H', \mathcal{U}')$ by shifting a edge as in (2) does not display all blocking pairs either, a contradiction. □

5.3 The proof of lemmas 49 and 50

Let $\vec{H}$ be an LR-graph and let X_1 and X_2 be LR-separators. We say that X_1 and X_2 *cross* if $X_1 \setminus X_2$ and $X_2 \setminus X_1$ are both non-empty. (Note that, $L \in X_1 \cap X_2$ and $R \notin X_1 \cup X_2$.)

Lemma 51. *Let (G, Σ) be a nearly 3-connected signed graph with no blocking vertex. Let $\vec{H}$ be an LR-graph arising from (G, Σ). Let X_1 and X_2 be crossing LR-separators. Then there exist edges e_1 and e_2 of $E(\vec{H})$ such that $\{e_1, e_2\}$ is an edge cut of $\vec{H}$ separating L from R and, for $i = 1, 2$, $X_i = (X_1 \cap X_2) \cup \{e_i\}$. In particular, the intercepting pair $\mathcal{B}_{\vec{H}}(X_i)$ is skewed.*

Proof. Remark 1 implies that,

Claim. *There is no set $Y \subseteq E(\vec{H})$ with $L \in Y$, $R \notin Y$ and $|\mathcal{B}(Y)| \leq 1$.*

Let a_1, a_2, b_1, b_2 denote the vertices of $\vec{H}$ where, for $i = 1, 2$, $\mathcal{B}_{\vec{H}}(X_i) = \{a_{3-i}, b_i\}$. Define,

$$W^{\cap} := \mathcal{B}_{\vec{H}}(X_1 \cap X_2) \qquad \text{and} \qquad W^{\cup} := \mathcal{B}_{\vec{H}}(X_1 \cup X_2).$$

The Claim implies that $|W^{\cap}|, |W^{\cup}| \geq 2$ and we may assume that $W^{\cap} = \{a_1, b_1\}$ and $W^{\cup} = \{a_2, b_2\}$. Let $\big(H, (s_1, t_1, s_2, t_2)\big)$ be the U-graph corresponding to $\vec{H}$. By the Claim and Menger's theorem, there exist vertex disjoint $\{s_1, t_1\} - \{a_1, b_1\}$ paths P_1 and P_2 where P_1 has end a_1 and P_2 end b_1. Similarly, there exists vertex disjoint $\{s_2, t_2\} - \{a_2, b_2\}$ paths Q_1 and Q_2 where Q_1 has end a_2 and Q_2 has end b_2. Define $Z := X_1 \triangle X_2$. Note that, for $i = 1, 2$ there is no $a_1 - b_2$ path F in $H[Z] - \{a_2, b_1\}$ for otherwise, the $\{s_1, t_1\} - \{s_2, t_2\}$ walk $P_1 \cup F \cup Q_2$ avoids a_2, b_1, a contradiction as $\mathcal{B}_{\vec{H}}(X_1) = \{a_2, b_1\}$. Similarly, there is no $a_2 - b_1$ path F in $H[Z] - \{a_1, b_2\}$. The Claim implies that there exists an $a_1 - a_2$ path F_1 in $H[Z]$. The component of $H[Z]$ containing F_1 does not contain either of b_1, b_2. Hence, by Lemma 33 it consists of a single edge $e_1 = (a_1, a_2)$. Similarly, there exists a $b_1 - b_2$ path F_2 of $H[Z]$ that consists of a single edge $e_2 = (b_1, b_2)$ and the component of $H[Z]$ that contains e_2 does not contain a_1, a_2. Hence, $Z = \{e_1, e_2\}$ and the result follows. $\square$

Proof of Lemma 49. Since (G, Σ) is not timid, there exists a U-graph $(H', \mathcal{U}')$ that arises from (G, Σ) and that displays all blocking pairs. Let $\vec{H}'$ be the LR-graph corresponding to that U-graph. Thus there exists $X \subseteq E(H)$, such that $\mathcal{B}_{\vec{H}'}(X \cup L) = \{\hat{s}, \hat{t}\}$, where s and t are the vertices of G corresponding to $\hat{s}$ and $\hat{t}$ respectively. (See definition of "corresponding vertex" in Section 1.2.)

Consider first the case where $\hat{s}, \hat{t}$ are not skewed in $\vec{H}'$. Let $(H, \mathcal{U})$ be the U-graph obtained from $(H', \mathcal{U}')$ by shifting X. The proof of Remark 35 implies that $(H, \mathcal{U})$ is obtained by unfolding (G, Σ') on s, t where $\Sigma' \subseteq \delta_G(s) \cup \delta_G(t)$ is a signature of (G, Σ). We claim that every blocking pair of (G, Σ) is displayed by the U-graph $(H, \mathcal{U})$. Consider an arbitrary blocking pair $\{c, d\}$ of (G, Σ). Then for some $Y \subseteq E(H)$, $\mathcal{B}_{\vec{H}'}(Y \cup \{L\}) = \{\hat{c}, \hat{d}\}$, where c and d are the vertices of G corresponding to $\hat{c}$ and $\hat{d}$ respectively. Note, that X and Y do not cross in H', for otherwise, Lemma 51 implies that $\hat{s}, \hat{t}$ are skewed in $\vec{H}'$, a contradiction. Thus either $Y \subseteq X$ or $Y \subseteq \bar{X}$. In the former case, $\bar{X} \cup Y$ displays the blocking pair c, d in $(H, \mathcal{U})$. In the latter case, $Y \setminus X$ displays the blocking pair c, d in $(H, \mathcal{U})$.

Consider now the case where $\hat{s}, \hat{t}$ are skewed in $\vec{H}'$, i.e., after possibly interchanging the labels of $\hat{s}$ and $\hat{t}$, there exist edges e, f of $E(H')$ such that $\{e, f\}$ is an edge cut of $\vec{H}'$ separating L from R, $e \in X$, $f \notin X$ and where $\hat{s}$ is an endpoint of e and $\hat{t}$ is an endpoint of f. Denote by $\hat{s}'$ the end of edge e in H' that is distinct from $\hat{s}$. Note, that $\hat{s}'$ and $\hat{t}$ are not skewed, for otherwise, either e or f is in series with another edge of H, contradicting Lemma 33. Note, $\mathcal{B}_{\vec{H}'}\big((X \setminus \{e\}) \cup L\big) = \{\hat{s}', \hat{t}\}$. Let $(H'', \mathcal{U}'')$ be the U-graph obtained from $(H', \mathcal{U}')$ by shifting $X \setminus \{e\}$. By the same argument as above the U-graph $(H'', \mathcal{U}'')$ displays all blocking pairs of (G, Σ). Finally, observe that $(H, \mathcal{U})$ is obtained from $(H'', \mathcal{U}'')$ by shifting $E(G) \setminus \{e\}$, or equivalently, up to equivalence, by shifting the edge e. $\square$

Lemma 52. *Let (G, Σ) be a nearly 3-connected signed graph with no blocking vertex. Let $\{s, t\}$ be a blocking pair that is not special. For $i = 1, 2$, let $\Gamma_i \subseteq \delta(s) \cup \delta(t)$ be a signature of (G, Σ). If $\Gamma_1 \triangle \Gamma_2$ is non-empty it is equal to one of $\delta(s)$, $\delta(t)$, or $\delta(s) \triangle \delta(t)$.*

Proof. Since Γ_1 and Γ_2 are signatures, $\Gamma_1 \triangle \Gamma_2 = \delta(U)$ for some $U \subseteq V(G)$. We may assume that $\delta(U)$ is distinct from $\emptyset$, $\delta(s)$, $\delta(t)$ and $\delta(s) \triangle \delta(t)$. We may assume that $s \notin U$ for otherwise we can replace U by $U \setminus \{s\}$ as $\delta(U) \triangle \delta(s) = \delta(U \triangle \{s\}) = \delta(U \setminus \{s\})$. Similarly, we may assume that $t \notin U$. Let $X := E(U) \cup \{(s, u) \in E(G) : u \in U\} \cup \{(t, u) \in E(G) : u \in U\}$. Then $\mathcal{B}(X) \subseteq \{s, t\}$ and $\mathcal{I}(X), \mathcal{I}(\bar{X})$ are both non-empty. Moreover, since (G, Σ) is nearly 3-connected, $\mathcal{B}(X) = \{s, t\}$. Hence, X is special, a contradiction. $\square$

Proof of Lemma 50. Let $(H', \mathcal{U}')$ be the U-graph obtained by unfolding (G, Σ') on s, t. It follows from Lemma 52 that $\Sigma \triangle \Sigma'$ is equal to one of (a) $\delta_G(s)$, (b) $\delta_G(t)$ or (c) $\delta_G(s) \triangle \delta_G(t)$. In all cases $H = H'$. For (a) $\mathcal{U}' = (s_2, t_1, s_1, t_2)$. For (b) $\mathcal{U}' = (s_1, t_2, s_2, t_1)$ and $(H, \mathcal{U}')$ is equivalent to $\big(H, (s_2, t_1, s_1, t_2)\big)$. For (c) $\mathcal{U}' = (s_2, t_2, s_1, t_1)$ and $(H, \mathcal{U}')$ is equivalent to $\big(H, (s_1, t_1, s_2, t_2)\big)$. $\square$

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