

Packing cycles with modularity constraints

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Abstract

We prove that for all positive integers k , there exists an integer $N = N(k)$ such that the following holds. Let G be a graph and let Γ an abelian group with no element of order two. Let $\gamma : E(G) \rightarrow \Gamma$ be a function mapping elements of Γ to the edges of G . A *non-zero cycle* is a cycle C such that $\sum_{e \in E(C)} \gamma(e) \neq 0$ where 0 is the identity element of Γ . Then G either contains k vertex disjoint non-zero cycles or there exists a set $X \subseteq V(G)$ with $|X| \leq N(k)$ such that $G - X$ contains no non-zero cycle.

An immediate consequence is that for all positive odd integers m , a graph G either contains k vertex disjoint cycles of length not congruent to $0 \pmod m$, or there exists a set X of vertices with $|X| \leq N(k)$ such that every cycle of $G - X$ has length congruent to $0 \pmod m$. No such value $N(k)$ exists when m is allowed to be even, as examples due to Reed and Thomassen show.

Key Words : Graph, Group labeling, Disjoint cycles, Erdős-Pósa Property

1 Introduction

Erdős and Pósa proved in [3] that there exists a function $f(k)$ such that for all positive integers k , either a graph contains k vertex disjoint cycles or there exists a set X of $f(k)$ vertices such that every cycle contains some vertex of X . We now say that a family of graphs \mathcal{F} has the *Erdős-Pósa property* if there exists a function $f(k)$ such that every graph G either has k vertex disjoint subgraphs in \mathcal{F} or there exists a set X of at most $f(k)$ vertices such that every subgraph of G isomorphic to a graph in \mathcal{F} contains a vertex in X . Such a function f will be referred to as an *Erdős-Pósa function* for the family \mathcal{F} , and such a set X of vertices intersecting every subgraph in \mathcal{F} is a *hitting set*.

An example of a family of graphs for which the Erdős-Pósa property does not hold is the class \mathcal{C}_{odd} , the set of cycles of odd length. This was first observed by Thomassen in [16]. Consider the graph constructed as follows. Start with the $k \times k$ grid for k odd with the vertices labeled $v_{i,j}$ for $1 \leq i \leq k$, $1 \leq j \leq k$, in the natural way. Add the edges $v_{1,i}v_{k,k-i+1}$ for $1 \leq i \leq k$ to form a projective planar graph G_k . The graph G_k does not contain two disjoint odd cycles, but there do not exist $\lfloor k/2 \rfloor$ vertices intersecting every odd cycle. Reed [13] gave a related construction called *Escher walls*. Furthermore, Reed [13] gives a partial characterization of when a graph has neither many disjoint odd cycles nor a small hitting set for the set of odd cycles by showing that such graphs

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must contain in a specific sense a large Escher wall. An immediate consequence of this result is that the Erdős-Pósa property holds for odd cycles in graphs embedded in any fixed orientable surface. Kawarabayashi and Nakamoto also independently showed that the Erdős-Pósa property holds for odd cycles in graphs embedded in any fixed orientable surface in [7]. In an alternate approach to the Erdős-Pósa property for odd cycles, Thomassen [17] showed that there exists a function $f(k)$ such that every $f(k)$ -connected graph either contains k disjoint odd cycles, or there exists a set X of at most $2k - 2$ vertices hitting all odd cycles. Rautenbach and Reed proved that the function $f(k)$ can be chosen to be linear in k in [12]. See [8] and [9] for further improvements on the connectivity bound.

In [2], Djeter and Neumann-lara give infinitely many pairs l and m such that the Erdős-Pósa property does not hold for the set of cycles of length $l \bmod m$. They propose the question of classifying for which values l and m does the family of cycles of length $l \bmod m$ have the Erdős-Pósa property. The projective planar graph G_k defined in the previous paragraph can be modified to show that the Erdős-Pósa does not hold for the class of cycles of length $l \bmod m$ for all fixed even m and $0 < l < m$ such that l is odd. To see this, fix such m and l and subdivide every edge of the grid to form a path of length m and subdivide every edge of the form $v_{1,i}v_{k,k-i+1}$ to form a path of length l . In the subdivided graph, every cycle of length $l \bmod m$ must use an odd number of the subpaths of length l . It follows that there do not exist even two disjoint cycles of length $l \bmod m$, and yet, there do not exist $\lfloor k/2 \rfloor$ vertices intersecting every cycle of length $l \bmod m$. Thomassen showed that for all positive integers m , the class of cycles of length $0 \pmod{m}$ does have the Erdős-Pósa property in [16]. We give the following theorem.

Theorem 1.1 *There exists a function $f(k)$ such that the following holds. For all positive integers k and all positive odd integers m , a graph G either has k disjoint cycles of non-zero length $\bmod m$, or there exists a set $X \subseteq V(G)$ of size at most $f(k)$ hitting all such cycles.*

As above, the example G_k can be modified to show that the family of cycles of non-zero length $\bmod m$ for a fixed even integer m does not have the Erdős-Pósa property. To see this, we subdivide each edge of the grid to be a path of length m and we subdivide the edges of the form $v_{1,i}v_{k,k-i+1}$ to form a path of length $m/2$. Such a graph does not contain two disjoint cycles of non-zero length $\bmod m$, nor does it contain a small hitting set for all such cycles.

One might object that the construction is only 2-connected and has many vertices of degree two. However, adding a single vertex v adjacent to every other vertex of the subdivided G_k yields a 3-connected graph that again serves as a counter-example to the Erdős-Pósa property holding for both the family of non-zero cycles $\bmod m$ when m is even and the family of cycles of length $l \bmod m$ for m even and $0 < l < m$, l odd.

To prove Theorem 1.1, we will actually prove a more general statement concerning group labeled graphs. Let Γ be an abelian group. As we will be considering abelian groups in this article, we will use the addition notation for groups with 0 indicating the identity element of a group. A Γ -labeling of a graph G is a function $\gamma : E(G) \rightarrow \Gamma$. For any subgraph H of G , we let $\gamma(H) = \sum_{e \in E(H)} \gamma(e)$. A *non-zero cycle* in a Γ -labeled graph is a cycle C such that $\gamma(C) \neq 0$. Recall that the *order* of any element $\alpha \neq 0$ of a group Γ is the minimum n such that $n\alpha = 0$. We prove the following.

Theorem 1.2 *There exist constants c and c' such that the following holds. Let G be a Γ -labeled graph where Γ does not have any elements of order two. Then for all positive integers k , either G*

contains k disjoint non-zero cycles, or there exists a set $X \subseteq V(G)$ with $|X| \leq c^{k^c}$ such that $G - X$ does not contain any non-zero cycles.

Theorem 1.1 is an immediate consequence of Theorem 1.2 as we can consider cycles of non-zero length mod m to be non-zero cycles in the graph labeled by the group \mathbb{Z}_m with every edge having weight one.

We observe that the condition that Γ be an abelian group is necessary; otherwise the idea of non-zero cycles is not well defined. Consider a graph labeled with elements from a non-abelian group Γ . Let C be a 3-cycle with edges e_1, e_2 , and e_3 labeled α, β , and $\alpha^{-1}\beta^{-1}$, respectively. If we calculate the weight of the subgraph traversing the edges e_1, e_3, e_2 in that order, then the weight of the cycle would be equal to the identity. However, if we calculate the weight of the cycle traversing the edges e_1, e_2, e_3 , the weight would be $\alpha\beta\alpha^{-1}\beta^{-1}$ which need not be equal to the identity.

We will need the following basic definitions. A *linkage* is a graph where every component is a path, and we say that a path *links* its endpoints. If A is a subset of the vertices of a graph G , then a non-trivial path intersecting A in exactly its endpoints is an A -*path*. If P is a path with endpoints x and y , the *interior* of the path, denoted $\text{int}(P)$, is the vertex set $V(P) - \{x, y\}$. Given two vertices a and b in a path P , the subpath of P linking a and b is denoted aPb . If G and H are two graphs, we denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ by $G \cup H$. Similarly, we denote by $G \cap H$ the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$. For any graph G of minimum degree two, the *branch vertices* of G is the set of vertices of degree at least three. A *segment* of G is a non-trivial path P in G such that P intersects the branch vertices exactly in its endpoints.

As a technicality, we extend the definition of the Erdős-Pósa property to group labeled graphs in the natural way. If \mathcal{F} is a family of Γ -labeled graphs, then \mathcal{F} has the Erdős-Pósa property with Erdős-Pósa function f if for all $k \geq 1$, every Γ -labeled graph G either has k disjoint subgraphs isomorphic to a graph in \mathcal{F} , or there exists a set $X \subseteq V(G)$ such that $G - X$ contains no subgraph isomorphic to a graph in \mathcal{F} . Here the isomorphism must respect the group values assigned to the edges.

The proof of Theorem 1.2 proceeds as follows. By choosing a huge function as a potential Erdős-Pósa function, we can ensure that a minimal group labeled counterexample G must contain a subgraph H isomorphic to a subdivision of a large grid-like graph called a wall. Moreover, the subgraph H is highly linked to every non-zero cycle of G . That is, if we let X be the vertices of H of \deg_H at least three, there does not exist a small cut separating X from any non-zero cycle C . This argument is laid out in Section 2. A result of [18] implies that there exists a collection of disjoint paths P_1, \dots, P_m for some large integer m , such that each P_i intersects X exactly in its endpoints, and, moreover, every path P_i has non-zero weight. The remainder of the argument consists in proving the claim, stated as Theorem 3.1, that if m is sufficiently large with respect to k , then the wall along with the non-zero paths P_1, \dots, P_m must contain k disjoint non-zero cycles. The proof of Theorem 1.2 (assuming Theorem 3.1) is given in Section 3. The proof of Theorem 3.1 comprises the main difficulties of the argument. After some preliminary lemmas presented in Section 4, the proof of Theorem 3.1 is given in Section 5.

2 Tangles and the Erdős-Pósa property

The proof of Theorem 1.2 relies heavily on the idea of a tangle, introduced in [14]. A *separation* of a graph G is a pair of subgraphs (A, B) such that $A \cup B = G$. The *order* of a separation (A, B) is

$|A \cap B|$. If $t \geq 1$ is a positive integer, a *tangle of order t* in a graph G is a set \mathcal{T} of ordered pairs such that

1. if (A, B) is a separation of G of order $< t$, then \mathcal{T} contains one of (A, B) , (B, A) ,
2. if (A_1, B_1) , (A_2, B_2) , and $(A_3, B_3) \in \mathcal{T}$, then $A_1 \cup A_2 \cup A_3 \neq G$,
3. and if $(A, B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

Tangles arise in a natural way when considering the Erdős-Pósa property for a given family of connected graphs.

Let \mathcal{F} be a family of graphs, possibly group labeled, and let f be a function. The pair (G, k) forms a *counterexample* to f being an Erdős-Pósa function for \mathcal{F} if the graph G neither contains k disjoint subgraphs each isomorphic to some graph in \mathcal{F} nor contains a hitting set $X \subseteq V(G)$ for the family \mathcal{F} with $|X| \leq f(k)$. If \mathcal{F} is a family of Γ -labeled graphs for some group Γ , then G must also be a Γ -labeled graph.

Lemma 2.1 *Let \mathcal{F} be a family of connected graphs, possibly group labeled. Let f be a positive function that is not an Erdős-Pósa function for \mathcal{F} , and let (G, k) be a counter-example to f being an Erdős-Pósa function for \mathcal{F} with k chosen minimal over all such counter-examples. Let t be a positive integer such that $t \leq f(k) - 2f(k-1)$ and $t \leq f(k)/3$, and let \mathcal{T} be the set of all ordered pairs (A, B) where (A, B) is a separation of order at most t such that B contains a subgraph in \mathcal{F} . Then \mathcal{T} forms a tangle in G of order t .*

Proof. We first claim that for any separation (A, B) of order less than t in G , at most one of A and B can contain a subgraph contained in \mathcal{F} . Assume, to reach a contradiction, that both A and B contain a subgraph in \mathcal{F} . Lest G contain k disjoint subgraphs in \mathcal{F} , there do not exist $k-1$ disjoint subgraphs in \mathcal{F} in either the graph $A - V(B)$ or the graph $B - V(A)$. It follows by our choice of (G, k) to minimize k that there exists a set $X_A \subseteq V(A)$ with $|X_A| \leq f(k-1)$ and a set $X_B \subseteq V(B)$ with $|X_B| \leq f(k-1)$ such that every subgraph of \mathcal{F} contained in G intersects the set $X := X_A \cup X_B \cup (V(A) \cap V(B))$. Moreover, $|X| \leq f(k)$, a contradiction to our choice of k and G . We conclude that if $(A, B) \in \mathcal{T}$, then $(B, A) \notin \mathcal{T}$. Note that this argument relies on our choice of \mathcal{F} to only contain connected graphs. Also, if (A, B) is a separation of order at most t , then at least one of A and B must contain a subgraph of \mathcal{F} by our choice of (G, k) to form a counter-example. Thus 1. is satisfied in the definition of tangle.

Property 2. and 3. in the definition of tangle are satisfied by our assumption that $t \leq f(k)/3$ and the fact that A cannot contain any subgraph of \mathcal{F} for any separation (A, B) in \mathcal{T} . ■

We say that G contains a $g \times g$ *grid minor* if there exist pairwise disjoint sets $\{X_{i,j} : 1 \leq i \leq g, 1 \leq j \leq g\}$ and edges $\{e_{i,j}^{right} \in E(G) : 1 \leq i \leq g, 1 \leq j \leq g-1\} \cup \{e_{i,j}^{down} : 1 \leq i \leq g-1, 1 \leq j \leq g\}$ such that $G[X_{i,j}]$ is connected, the edge $e_{i,j}^{right}$ has one end in $X_{i,j}$ and one end in $X_{i,j+1}$, and the edge $e_{i,j}^{down}$ has one end in $X_{i,j}$ and one end in $X_{i+1,j}$. Let H be the subgraph of G comprising a $g \times g$ grid minor. The i^{th} row of H is the set of edges $\{e_{i,j}^{right} : 1 \leq j \leq g-1\}$ for some fixed i . Let \mathcal{T}_H be the set of all separations (A, B) of order $< g$ such that $E(B)$ includes a row of H . It follows from a result of [14] (Theorem 7.3) that \mathcal{T}_H is a tangle in G of order g . A tangle \mathcal{T} *dominates* H if $\mathcal{T}_H \subseteq \mathcal{T}$.

The following theorem of Robertson, Seymour, and Thomas [15] gives an improved explicit bound to a similar qualitative result appearing in [14].

Theorem 2.2 *Let $g \geq 2$ and let \mathcal{T} be a tangle in G of order $\geq 20^{g^4(2g-1)}$. Then \mathcal{T} dominates a $g \times g$ grid minor.*

Combining Theorem 2.2 with Lemma 2.1, we will see that if we are looking for an Erdős-Pósa function for a particular family \mathcal{F} of graphs, then an appropriately minimal counter-example G will not only have a large grid minor, but that grid minor will be well connected to every subgraph of \mathcal{F} contained in G . We make this idea explicit below as Theorem 2.3.

For the proofs to come, we will work with a variation of a grid minor that will allow us to consider a subdivision instead of a minor. We define the graph $W_{l,m}$ the $l \times m$ wall. Let $V(W_{l,m}) = \{v_{i,j} : 1 \leq i \leq l, 1 \leq j \leq m\}$ and

$$\begin{aligned} E(W_{l,m}) = & \{v_{i,j}v_{i,j+1} : 1 \leq i \leq l, 1 \leq j \leq m-1\} \\ & \cup \{v_{2i-1,2j-1}v_{2i,2j-1} : 1 \leq i \leq l/2, 1 \leq j \leq \lceil m/2 \rceil\} \\ & \cup \{v_{2i,2j}v_{2i+1,2j} : 1 \leq i \leq l-1/2, 1 \leq j \leq m/2\} \end{aligned}$$

Equivalently, $W_{l,m}$ consists of l disjoint paths P_1, \dots, P_l each containing m vertices such that for odd i , the odd vertices of P_i and P_{i+1} are adjacent, and for even i , the even vertices of P_i and P_{i+1} are adjacent. It is easy to see that if a graph G contains a $g \times g$ grid minor, then G contains a subgraph isomorphic to a subdivision of the $g \times g$ wall. Moreover, the subdivision contains every row of the grid minor.

Theorem 2.3 *Let \mathcal{F} be a family of connected, possibly group labeled, graphs. For all positive integers k , let $l = l(k)$, $m = m(k)$, and $t = t(k)$ be positive integers satisfying $l(k) \geq t(k)$ and $m(k) \geq t(k)$. For all k , let $\alpha(k) = \max\{m(k), t(k) + 1, l(k)\}$. Define the function*

$$f(k) = 4^k 20^{\alpha(k)^4(2\alpha(k)-1)}.$$

Assume f is not an Erdős-Pósa function for \mathcal{F} , and let (G, k) be a counter-example with k chosen minimal among all such counter-examples. Then there exists a subgraph H of G isomorphic to a subdivision of the $l \times m$ wall such that for every graph F of G isomorphic to a graph of \mathcal{F} , there exist t vertex disjoint paths linking the branch vertices of H and $V(F)$ in G .

Proof. Let \mathcal{F} , l , m , t , α , and f be given. Let (G, k) be a counter-example to f being an Erdős-Pósa function for the family \mathcal{F} , and assume it is chosen from all such counter-examples to minimize k . Let \mathcal{T} be all the separations (A, B) of order at most $20^{\alpha(k)^4(2\alpha(k)-1)}$ such that B contains a subgraph in \mathcal{F} . Lemma 2.1 implies that \mathcal{T} is a tangle of order $20^{\alpha(k)^4(2\alpha(k)-1)}$. Theorem 2.2 implies that there exists an $\alpha \times \alpha$ grid minor H such that \mathcal{T} dominates H . As we observed above, H contains a subgraph H' isomorphic to a subdivision of the $l \times m$ wall. Let the branch vertices of H' be labeled $v_{i,j}$ in the natural way. We may assume that the path in H' linking $v_{i,j}$ and $v_{i,j+1}$ that corresponds to the edge $v_{i,j}v_{i,j+1}$ in the $l \times m$ wall contains exactly one edge of the i^{th} row of the $\alpha \times \alpha$ grid minor.

Let X be the vertices of H' of $\deg_{H'}$ at least three. Observe that if we let $X_{i,j}$ be as in the definition of the $\alpha \times \alpha$ grid minor, then every vertex $v_{i,j}$ of X is contained in $X_{i,j}$. It follows that for any row of H , if we let Y be the endpoints of the edges comprising the row, then there exist at least $t+1$ disjoint paths linking Y to X . Assume the theorem is false and that there exists a subgraph D of G contained in \mathcal{F} , but that there do not exist t disjoint paths linking D and X . Then there exists a separation (A, B) of order at most $t-1$ with $X \subseteq V(A)$ and $D \subseteq V(B)$. The separation (A, B) is

contained in \mathcal{T} , and consequently, in \mathcal{T}_H . It follows that there exists an index i such that the i^{th} a row of the $\alpha \times \alpha$ grid minor contained in B . But this is a contradiction, as we saw above, since there exist $t + 1$ disjoint paths linking the row and the set X . This completes the proof of the theorem. ■

3 Proof of Theorem 1.2

In the proof of Theorem 1.2, we utilize Theorem 2.3 to ensure that a potential counter-example to Theorem 1.2 has a subdivided wall which we can make as large as we need by choosing a sufficiently large Erdős-Pósa function. The next theorem allows us to find many non-zero cycles if we can find many non-zero paths attaching to the branch vertices of the wall.

Theorem 3.1 *There exists a constant c such that for all integers $k \geq 1$ the following is true. Let $n_k = ck^{87}$ and let $l \geq n_k$, $m \geq n_k$ be positive integers such that l is congruent to 0 mod 4 and m is congruent to 7 mod 8. Let Γ be an abelian group not containing any elements of order two, and let G be a Γ -labeled graph with weight function γ . Let H be a subgraph of G isomorphic to a subdivision of $W_{l,m}$ and let X be the vertices of H of degree at least three. If G contains n_k disjoint X -paths P_1, \dots, P_{n_k} such that $\gamma(P_i) \neq 0$ for $1 \leq i \leq n_k$, then G contains k vertex disjoint cycles of non-zero weight.*

The proof of Theorem 3.1 is somewhat technical, and we leave the proof for Section 5. Note that if l is even and m is odd, the $l \times m$ wall has minimum degree two. Throughout the proofs, assuming that a given wall has minimum degree two will simplify the notation and avoid the technicality of repeatedly deleting vertices of degree one. Thus, our choice of l and m to satisfy the given modularity constraints is merely a convenience, and not of fundamental importance to the results.

In order to find many such non-zero paths attaching to the branch vertices, we need the following result from [18].

Theorem 3.2 *There exists a constant c such that the following holds. Let Γ be an abelian group and let γ be a Γ -labeling of a graph G . Let $A \subseteq V(G)$ be a fixed set of vertices of G . Then either there exist disjoint A -paths P_1, P_2, \dots, P_k with $\gamma(P_i) \neq 0$, or there exists a set $X \subseteq V(G)$ with $|X| \leq ck^4$ such that $G - X$ has no non-zero A path.*

We are now reach to give the proof of Theorem 1.2.

Proof. (Theorem 1.2, assuming Theorem 3.1) Let c_1 be the constant in the statement of Theorem 3.2. Let c_2 be the constant in Theorem 3.1. We set $t(k) := c_1(c_2k^{87})^4 + 3$, $l(k) = 4\lceil t(k)/4 \rceil$ and $m(k) = 8\lceil t(k)/8 \rceil + 7$. We set $\alpha(k) = \max\{l(k), m(k), t(k) + 1\}$. Fix c and c' such that

$$c^{k^{c'}} \geq 4^k 20^{\alpha(k)^4(2\alpha(k)-1)}.$$

We claim c and c' satisfy the statement of Theorem 1.2. Assume not. Then there exists a group Γ with no element of order two, a graph G , a positive integer k , and a Γ -labeling γ of G such that G neither has k disjoint non-zero cycles, nor does there exist a covering set X of size at most $c^{k^{c'}}$. Assume that we chose G , Γ , and k to minimize the value of k . By Theorem 2.3, there exists a subgraph H of G isomorphic to a subdivision of the $l \times m$ wall such that there exist $t(k)$ disjoint paths linking the branch vertices of H to any non-zero cycle in G . Let X be the branch vertices of H . By Theorem 3.1, there do not exist c_2k^{87} disjoint non-zero X -paths. Consequently, by Theorem

3.2, there exists a set Z of vertices with $|Z| \leq c_1(c_2k^{87})^4$ such that $G - Z$ does not contain any non-zero X -path.

We claim the set Z intersects every non-zero cycle in G . This will contradict our choice of G to be a counter-example, as $|Z| \leq c^{k^{c'}}$. If Z is not a hitting set, then there exists a non-zero cycle C disjoint from Z . There exist $t(k)$ disjoint paths linking X to C . It follows that there exist three disjoint paths from C to X avoiding Z , call them P_1, P_2 , and P_3 . We claim that $P_1 \cup P_2 \cup P_3 \cup C$ contains a non-zero X -path, contrary to the fact that Z intersects every such non-zero X -path. To see this, for $i = 1, 2, 3$, let Q_i be the subpath in C linking the endpoints of P_i and P_{i+1} avoiding P_{i+2} where all the index notation is taken mod 3. Let $a_i = \gamma(P_i)$ for $i = 1, 2, 3$. Lest $P_i \cup Q_i \cup P_{i+1}$ contain a non-zero X -path, we see $\gamma(Q_i) = -a_i - a_{i+1}$ for $i = 1, 2, 3$. Adding the terms together, we get that $\gamma(C) = -2a_1 - 2a_2 - 2a_3$. Similarly, lest $P_i \cup Q_{i-1} \cup Q_{i+1} \cup P_{i+1}$ contain a non-zero X -path, we see that $\gamma(Q_{i-1}) + \gamma(Q_{i+1}) = \gamma(Q_i) = -a_i - a_{i+1}$ for $i = 1, 2, 3$. This implies that $\gamma(C) = -2a_i - 2a_{i+1}$. Merging the equations, we see that $-2a_i - 2a_{i+1} = -2a_1 - 2a_2 - 2a_3$ for $i = 1, 2, 3$. It follows that $2a_i = 0$ for $i = 1, 2, 3$. Consequently, $\gamma(C) = -2a_1 - 2a_2 - 2a_3 = 0$, a contradiction. We conclude that $P_1 \cup P_2 \cup P_3 \cup C$ in fact does contain a non-zero X -path. This final contradiction to our choice of Z completes the proof of the theorem. \blacksquare

4 Non-zero paths attaching to ladders and grids

We begin with several preliminary results before proceeding with the proof of Theorem 3.1 in the next section. Let G be a Γ -labeled graph with weight function γ for some group Γ . A *non-zero theta* is a subgraph of G consisting of three internally disjoint paths P_1, P_2, P_3 each with endpoints equal to x and y for some pair of vertices x, y and furthermore, such that there exists an index $i \in \{1, 2, 3\}$ such that $\gamma(P_i) \neq 0$.

Observation 1 *Let G be a Γ -labeled graph with weight function γ . If Γ does not contain an element of order two, then every non-zero theta contains a non-zero cycle.*

Proof. Let P_1, P_2, P_3, G and Γ be as in the statement. Without loss of generality, assume $\gamma(P_1) = \alpha$, an element of Γ which has order not equal to two. Lest $P_1 \cup P_2$ ($P_1 \cup P_3$) form a non-zero cycle, we see that $\gamma(P_2) = \gamma(P_3) = -\alpha$. By our choice of α , the graph $P_2 \cup P_3$ is a non-zero cycle, proving the observation. \blacksquare

We will need the following classic result of Erdős and Szekeres.

Theorem 4.1 [3] *Let $a = (a_1, a_2, \dots, a_n)$ be a sequence of positive, distinct integers. If $n \geq k^2$, then a either contains a strictly increasing subsequence of length k or a strictly decreasing subsequence of length k .*

From Theorem 4.1, we conclude the following lemma.

Lemma 4.2 *Let P_1 and P_2 be two vertex disjoint paths in a Γ -labeled graph G with weight function γ . Assume that Γ does not contain any elements of order two. Let Q_1, \dots, Q_l be vertex disjoint paths in G such that $\gamma(Q_i)$ is non-zero and each Q_i has one endpoint in P_1 and one endpoint in P_2 for $i = 1, \dots, l$. If*

$$l \geq 9k^2,$$

then G contains k vertex disjoint non-zero cycles.

Proof. Let G, P_1, P_2, Γ , and Q_1, \dots, Q_l be as in the statement. We will prove that G contains k disjoint non-zero cycles by proving that G in fact contains k disjoint non-zero thetas. Label the vertices of $P_1 \cap (\bigcup_i V(Q_i))$ by $1, 2, \dots, n$ so that they occur on the path P_1 in that order, and similarly, let the vertices of $P_2 \cap (\bigcup_i V(Q_i))$ be labeled $1', 2', \dots, n'$ so that they occur in that order on the path P_2 . We define a sequence $a(1), a(2), \dots, a(n)$ such that for all $i, 1 \leq i \leq n$, there exists an index j such that Q_j has endpoints i in P_1 and $a(i)'$ in P_2 . By Theorem 4.1, there exists values $1 \leq \pi_1 < \pi_2 < \dots < \pi_{3k} \leq n$ such that the subsequence $a(\pi_1), a(\pi_2), \dots, a(\pi_{3k})$ is either monotone increasing or decreasing. Let $\sigma_1, \sigma_2, \dots, \sigma_{3k}$ be such that Q_{σ_i} has endpoints π_i and $a(\pi_i)'$ for $1 \leq i \leq 3k$. Then for all $t, 2 \leq t \leq 3k - 1$, the subgraph $\pi_{t-1}P_1\pi_{t+1} \cup a(\pi_{t-1})'P_2a(\pi_{t+1})' \cup Q_{\sigma_{t-1}} \cup Q_{\sigma_t} \cup Q_{\sigma_{t+1}}$ forms a non-zero theta. Thus we have constructed k distinct non-zero thetas and they will be disjoint by the fact that the subsequence $a(\pi_1), a(\pi_2), \dots, a(\pi_{3k})$ is monotone. ■

The following related lemma is an immediate consequence of a theorem of Rado [11] concerning sets of intervals.

Lemma 4.3 [11] *Let I_1, I_2, \dots, I_{k^2} be subintervals of $[0, 1]$. Then either there exists a subcollection of k pairwise disjoint intervals or k pairwise intersecting intervals.*

Lemma 4.4 follows easily from Lemma 4.3. We omit the proof here.

Lemma 4.4 *Let P be a path and let Q_1, \dots, Q_n be disjoint $V(P)$ -paths. If $n \geq k^2$, then one of the following outcomes holds.*

1. *There exists a subset $I \subseteq \{1, 2, \dots, n\}$ with $|I| \geq k$ and an edge $e \in V(P)$ such that for every $i \in I$, the path Q_i has one endpoint in each of the two components of $P - e$, or*
2. *there exists a subset $I \subseteq \{1, 2, \dots, n\}$ with $|I| \geq k$ such that for every distinct $i, j \in I$, the subpath of P linking the ends of Q_i is disjoint from the subpath of P linking the ends of Q_j .*

Before proceeding to analyze the situation when we have many non-zero paths attaching to a subdivision of a grid, we first consider a simpler graph. The k ladder, denoted L_k is the graph with vertex set $\{v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_k\}$ and edges $\{v_i v_{i+1} : 1 \leq i \leq k - 1\} \cup \{u_i u_{i+1} : 1 \leq i \leq k - 1\} \cup \{u_i v_i : 1 \leq i \leq k\}$. In other words, the k ladder is formed by taking two paths of length $k - 1$ and joining the corresponding vertices on the paths with an edge.

Lemma 4.5 *Let G be a Γ -labeled graph with weight function γ . Assume Γ does not contain any elements of order two. Let H be a subgraph of G isomorphic to a subdivision of an l -ladder for $l \geq 3$. Let X be the branch vertices of H . Let P_1, \dots, P_n be non-zero $V(H)$ -paths such that no P_i has both endpoints contained in a segment of H . For all positive integers k , if*

$$n \geq 27816k^6 \geq [3(9k^2)^2 + (3k + 2)] (6(18k^2 + 2)),$$

then G contains k vertex disjoint non-zero cycles.

Proof. Let G, H, Γ, γ , and P_1, \dots, P_n be as in the statement. Let X be the branch vertices of H . Assume the lemma is false, and let G be a counter-example.

First, assume there exists a segment P of H such that P contains the endpoints of $18k^2 + 2$ distinct paths P_i . Let Z be the internal vertices of P . There are then $18k^2$ distinct paths P_i with one endpoint in the set Z . Observe that there exist two (not necessarily disjoint) paths in $H - Z$ covering the vertices of $H - Z$. It follows from Lemma 4.2 that there exists a path P' in $H - Z$ such for at least $9k^2$ distinct indices i , the path P_i has one endpoint in P and one endpoint in P' . Lemma 4.2 implies that G contains k vertex disjoint cycles, a contradiction to our choice of G .

We conclude that there exists a subset $I \subseteq \{1, 2, \dots, n\}$ such that for any $i, j \in I$, there does not exist a segment of H containing endpoints of both P_i and P_j . Since for every path P_i there are at most six distinct segments of H containing an endpoint of P_i , and combined with the arguments of the previous paragraph, we see that $|I| \geq n/(6(18k^2 + 2)) \geq [3(9k^2)^2 + (3k + 2)]$.

For the remainder of the proof, we will need to refer more specifically to the segments and branch vertices of X . The vertices of the l -ladder are labeled $\{u_1, u_2, \dots, u_l, v_1, v_2, \dots, v_l\}$. We naturally label the branch vertices X of H with $\{u_2, u_3, \dots, u_{l-1}, v_2, \dots, v_{l-1}\}$. We label the segments of H as follows. Let the segments Q_2, Q_3, \dots, Q_{l-2} be defined where Q_i has endpoints u_i and u_{i+1} for all $2 \leq i \leq l-2$. Let the segments R_2, R_3, \dots, R_{l-2} be defined where R_i has endpoints v_i and v_{i+1} for $2 \leq i \leq l-2$. Let S_1, S_2, \dots, S_l be defined where S_i has endpoints u_i and v_i for $2 \leq i \leq l-1$, S_1 has endpoints u_2 and v_2 , and S_l has endpoints u_{l-1} and v_{l-1} . Let the paths $Q = \bigcup_2^{l-2} Q_i$ and $R = \bigcup_2^{l-2} R_i$.

For every $i \in I$, we want to define a non-zero path \overline{P}_i extending P_i such that the endpoints of \overline{P}_i are contained in $R \cup Q$. Towards this end, for every $i \in I$, let x_i and y_i be the endpoints of P_i . If $x_i, y_i \in R \cup Q$, let $\overline{P}_i = P_i$. Lest G contain k disjoint non-zero thetas, there exists a set of at most $3k$ indices j , $2 \leq j \leq l-1$, such that S_j is a non-zero path. Let $J \subseteq \{2, 3, \dots, l-1\}$ such that S_j has weight zero for all $j \in J$. We have excluded 1 and l from the set J to ensure that the paths \overline{P}_i that we construct will be pairwise disjoint. If at least one endpoint of P_i is contained in a path S_j for some $j \in J$, then we construct \overline{P}_i as follows. Without loss of generality, assume x_i is contained in S_j for some $j \in J$. Since $\gamma(S_j) = 0$, one of the paths $u_j S_j x_i P_i y_i$ or $v_j S_j x_i P_i y_i$ has non-zero weight. To see this, if both have weight zero, then $\gamma(u_j S_j x_i) = \gamma(v_j S_j x_i) = -\gamma(P_i)$. But given that Γ does not contain elements of order two, this implies that $\gamma(S_j) \neq 0$ and contradicts our assumption that $j \in J$. If $y_i \in R \cup Q$, we let $\overline{P}_i = u_j S_j x_i P_i y_i$ if $\gamma(u_j S_j x_i P_i y_i) \neq 0$ and otherwise, let $\overline{P}_i = v_j S_j x_i P_i y_i$. If y_i is contained in $S_{j'}$ for some $j' \in J$, then by the same argument as above, at least one of the four paths $u_j S_j x_i P_i y_i S_{j'} u_{j'}$, $u_j S_j x_i P_i y_i S_{j'} v_{j'}$, $v_j S_j x_i P_i y_i S_{j'} u_{j'}$, or $v_j S_j x_i P_i y_i S_{j'} v_{j'}$ is a non-zero path. We set \overline{P}_i to such a non-zero path, arbitrarily choosing one if there is more than one such non-zero path. Let $I_1 \subseteq I$ be the set of indices for which \overline{P}_i is defined. By construction, $|I_1| \geq |I| - (3k + 2) \geq 3(9k^2)^2$. Also, by construction, \overline{P}_i and \overline{P}_j are disjoint for every distinct $i, j \in I_1$.

By Lemma 4.2, the path \overline{P}_i has one endpoint in Q and one endpoint in R for at most $9k^2$ distinct indices $i \in I_1$. Thus without loss of generality, we may assume that there exists a set $I_2 \subseteq I_1$ with $|I_2| \geq (|I_1| - 9k^2)/2 \geq (9k^2)^2$ such that for every index $i \in I_2$, the path \overline{P}_i has both endpoints in R . Let \overline{x}_i and \overline{y}_i be the endpoints of \overline{P}_i for all $i \in I_2$. Consider the subpaths of $\overline{x}_i R \overline{y}_i$ for all $i \in I_2$. By Lemma 4.4, either there exists an edge of R intersecting at least $9k^2$ of the subpaths $\overline{x}_i R \overline{y}_i$ or there exists $9k^2$ disjoint such subpaths. In the first case, say the edge e intersects at least $9k^2$ of the subpaths $\overline{x}_i R \overline{y}_i$. There exist $9k^2$ disjoint non-zero paths, each with one endpoint in each of two paths of $R - e$. Lemma 4.2 then contradicts our choice of G to not contain k vertex disjoint non-zero cycles.

In the other possible outcome of the application of Lemma 4.4, there exist $9k^2$ distinct indices $i \in I_2$ such that the subpaths $\overline{x_i R y_i}$ are pairwise disjoint. In fact, we will only need $2k$ such paths to derive a contradiction. For each such i , without loss of generality, assume that the vertices $v_2, \overline{x_i}, \overline{y_i}, v_{l-1}$ occur on the path R in that order. Let $\pi_L(i)$ be the index in $\{2, \dots, l-1\}$ such that $v_{\pi_L(i)}$ is as close to $\overline{x_i}$ as possible on the subpath $v_2 R \overline{x_i}$. Similarly, let $\pi_R(i)$ be the index such that $v_{\pi_R(i)}$ is as close as possible to $\overline{y_i}$ on the subpath $\overline{y_i} R v_{l-1}$. Note that $v_{\pi_L(i)}$ may in fact be equal to $\overline{x_i}$, and, symmetrically, $v_{\pi_R(i)}$ may be equal to $\overline{y_i}$. Given $2k$ indices such that the subpaths $\overline{x_i R y_i}$ are pairwise disjoint, there exists a subset $I_3 \subseteq I_2$ of k indices i such that $v_{\pi_R(i)} \neq v_{\pi_L(j)}$ for all distinct $i, j \in I_3$. Equivalently, we see that the subpath $v_{\pi_L(i)} R v_{\pi_R(i)}$ is disjoint from $v_{\pi_L(j)} R v_{\pi_R(j)}$ for all $i \neq j$. For all $i \in I_3$, by the construction of the path $\overline{P_i}$, the path P_i has both endpoints contained in the cycle $C_i := v_{\pi_L(i)} S_{\pi_L(i)} u_{\pi_L(i)} Q u_{\pi_R(i)} S_{\pi_R(i)} v_{\pi_R(i)} R v_{\pi_L(i)}$. We conclude that $P_i \cup C_i$ forms a non-zero theta for each $i \in I_3$, and consequently, G contains k vertex disjoint non-zero thetas. The observation then implies G contains k vertex disjoint non-zero cycles, contradicting our choice of G . This completes the proof of the lemma. \blacksquare

Throughout the following lemmas, we will often consider a graph G isomorphic to a subdivision of $W_{l,m}$ the $l \times m$ wall. We will include the assumption that l is even and m is odd to ensure that the graph has minimum degree two. Every branch vertex corresponds naturally to a vertex of the wall $W_{l,m}$. We will refer to the labeling of the set X of branch vertices so that

$$X = \{v_{i,j} : 2 \leq i \leq l-1, 2 \leq j \leq m-1\} \cup \{v_{1,i} : 2 \leq i \leq m-1, i \text{ odd}\} \cup \{v_{l,i} : 2 \leq i \leq m-1, i \text{ odd}\}$$

as the *canonical labeling* of the branch vertices X . Let P be a segment of G with end points $v_{i,j}$ and $v_{i',j'}$ for some i, i', j, j' . If $i = i'$, we say P is a *horizontal segment* and otherwise say P is a *vertical segment*.

Lemma 4.6 *Let G be a Γ -labeled graph with weight function γ . Assume G is isomorphic to a subdivision of $W_{l,m}$ with l even and m odd. Also assume that Γ does not contain any elements of order two. If G has at least $24k^2$ non-zero segments, then G contains k disjoint non-zero cycles.*

Proof. Let G, Γ , and γ be as in the statement. Let X be the set of branch vertices and have the canonical labeling. Let the path $S_i, 1 \leq i \leq l$, be the union of every horizontal segment with end points $v_{i,j}$ and $v_{i,j+1}$ for some index j .

First, we will see that the lemma holds if G has at least $12k^2$ vertical non-zero segments. If there exists an index i such that there exist $3k$ non-zero vertical segments with one endpoint in each of the paths S_i and S_{i+1} , then G contains k vertex disjoint non-zero thetas. Observation 1 implies that G then contains k vertex disjoint non-zero cycles. Notice that for any index $i, 1 \leq i \leq l-1$, any non-zero vertical segment with endpoints in S_i and S_{i+1} is contained in a non-zero theta using vertices of S_{i-1}, S_i, S_{i+1} , and S_{i+2} as well as vertical segments with their endpoints contained in S_{i-1}, S_i, S_{i+1} , and S_{i+2} . Note that the paths S_{i-1} and S_{i+2} may be necessary for to find a non-zero theta containing the first or last vertical segment on the outside boundary of H . It follows that if there exist $4k$ distinct indices i with $1 \leq i \leq l-1$, such that there exists a non-zero vertical segment attaching to S_i and S_{i+1} , then G contains k vertex disjoint non-zero thetas. We conclude that if there exist at least $(3k)(4k) = 12k^2$ non-zero vertical segments, then the conclusion of the lemma is satisfied.

We will now show that if G has at least $12k^2$ non-zero horizontal segments, then the lemma holds as well. For any index i , $1 \leq i \leq l$, we can naturally order the horizontal segments on S_i by the order in which they occur when traversing the path S_i . Any non-zero horizontal segment is contained in a non-zero theta that intersects S_i in at most the previous and the next segments on S_i . By choosing such non-zero thetas to contain as few vertices as possible, it follows that if S_i contains at least $4k$ non-zero segments, then G contains k disjoint non-zero thetas and the lemma holds. For all i , $2 \leq i \leq l - 1$, every non-zero horizontal segment on S_i is contained in a non-zero theta using only vertices of S_{i-1} , S_i , and S_{i+1} as well as vertical segments attaching to S_{i-1} , S_i , and S_{i+1} . Similarly, when $i \in \{1, l\}$ and there exists a non-zero segment on S_i then there exists a non-zero theta contained in S_1, S_2, S_3 or S_l, S_{l-1}, S_{l-2} as well as the vertical segments attaching to them. We conclude that if there exist at least $3k$ distinct indices such that S_i contains a non-zero horizontal segment, then G contains k vertex disjoint non-zero cycles. It follows that if G contains at least $(3k)(4k) \leq 12k^2$ non-zero horizontal segments, then G contains k disjoint non-zero cycles.

It follows that if G contains at least $24k^2$ non-zero segments, then G contains k disjoint non-zero cycles and the lemma is proven. \blacksquare

Lemma 4.7 *Let G be a Γ -labeled graph with weight function γ . Let l and m be integers, $l \geq 4$, $m \geq 5$, with l even and m odd. Let H be a subgraph of G isomorphic to a subdivision of $W_{l,m}$. Assume that Γ does not contain any elements of order two. Let k and n be integers with*

$$n \geq 9,320,850k^8 \geq 6(54k^2 + 2)(27816k^6 + (4k + 2)(81k^4) + 24k^2).$$

Let P_1, \dots, P_n be disjoint non-zero $V(H)$ -paths in G such that for all $1 \leq i \leq n$, the path P_i does not have both endpoints contained in a segment of H . Then G contains k disjoint non-zero cycles.

Proof. Let G, H, Γ, γ , and P_1, \dots, P_n be as in the statement. Let X be the branch vertices of H , and let X have the canonical labeling.

We first observe that if there exists a segment P of H and at least $54k^2 + 2$ distinct indices i such that P_i has an endpoint in P , then G contains k disjoint non-zero cycles. Observe that there exists a path in H containing every horizontal segment. The vertical segments can be covered by two (not necessarily disjoint) paths. If we let Y be the internal vertices of P , we see that there exists six paths covering the vertices of $H - Y$. It follows that there exists a subpath Q in H and at least $9k^2$ distinct indices i such that P_i has one endpoint on P and one endpoint in Q . By Lemma 4.2, G contains k disjoint non-zero cycles.

For every index i , the path P_i has an endpoint in at most six distinct segments of H , three at each endpoint. We conclude that we may assume there exists a set $I \subseteq \{1, 2, \dots, n\}$ of at least $n/6(54k^2 + 2) \geq 27816k^6 + (4k + 2)(81k^4) + 24k^2$ distinct indices such that P_i and P_j do not have endpoints contained in a common segment of H for all distinct $i, j \in I$. For all i , $2 \leq i \leq l - 1$, we let S_i be the union of every horizontal segment with endpoints equal to $v_{i,j}$ and $v_{i,j+1}$ for some index j . We let S_1 be formed from the ‘‘upper left’’ vertical segment with endpoints $v_{2,2}$ and $v_{1,3}$, the ‘‘upper right’’ vertical segment with endpoints $v_{1,m-2}$ and $v_{2,m-1}$, and every horizontal segment with endpoints $v_{1,j}$ and $v_{1,j+2}$ for some index j . Symmetrically, we let S_l be the union of the ‘‘lower left’’ vertical segment with endpoints $v_{l-1,2}$ and $v_{l,3}$, the ‘‘lower right’’ vertical segment with endpoints $v_{l-1,m-1}$ and $v_{l,m-2}$, and every horizontal segment with endpoints $v_{l,j}$ and $v_{l,j+2}$. We let Q be the union of the vertical segments with endpoints $v_{i,2}$ and $v_{i+1,2}$ for some index i , $2 \leq i \leq l - 2$.

Symmetrically, we let R be the union of the vertical segments with endpoints $v_{i,m-1}$ and $v_{i+1,m-1}$ for i , $2 \leq i \leq l-2$.

We would like to convert the set of disjoint $V(H)$ -paths $\{P_i : i \in I\}$ to a set of disjoint non-zero paths \overline{P}_i attaching to the subdivided ladder on the vertex set $Z := V(Q) \cup V(R) \cup \left(\bigcup_1^l V(S_i)\right)$. If P_i has both endpoints in Z , then let $\overline{P}_i = P_i$. For paths P_i such that exactly one endpoint of P_i is contained in a vertical segment T of H of weight zero with $T \not\subseteq Z$, observe that T has endpoints x and y in Z . Let z be the endpoint of P_i in Z and z' be the endpoint in T . Given that the segment T was assumed to have zero weight and the fact that Γ does not contain an element of order two, either the path $zP_iz'Tx$ or $zP_iz'Ty$ is a non-zero Z -path. We let $\overline{P}_i = zP_iz'Tx$ if $\gamma(zP_iz'Tx) \neq 0$ and set $\overline{P}_i = zP_iz'Ty$ otherwise. For $i \in I$ such that the endpoints of P_i are contained in exactly two vertical segments T_1 and T_2 with $\gamma(T_1) = \gamma(T_2) = 0$ and $T_1, T_2 \not\subseteq Z$, we let the endpoints of T_1 be x_1 and y_1 , and we let the endpoints of T_2 be x_2 and y_2 . At least one of the paths $x_1T_1zP_iz'T_2x_2$, $x_1T_1zP_iz'T_2y_2$, $y_1T_1zP_iz'T_2x_2$, or $y_1T_1zP_iz'T_2y_2$ must be a non-zero path. We choose one such non-zero path arbitrarily and assign it to be \overline{P}_i .

By Lemma 4.6, we may assume H has at most $24k^2$ non-zero segments. We conclude that there exists a set $I_1 \subseteq I$ such that \overline{P}_i is defined for all $i \in I_1$ with $|I_1| \geq |I| - 24k^2 \geq 27816k^6 + (4k+2)(81k^4)$. Observe that for all distinct $i, j \in I_1$, $V(\overline{P}_i) \cap V(\overline{P}_j) = \emptyset$ by construction.

We would like to apply Lemma 4.5. In order to do so, we need to find a large subset of $I_2 \subseteq I_1$ such that for any index $i \in I_2$, there does not exist an index j , $1 \leq j \leq l$, such that the path \overline{P}_i has both endpoints contained in S_j . This will suffice to ensure that no path \overline{P}_i has both endpoints in a single segment of the subdivided ladder on the vertex set Z , since by the construction no path \overline{P}_i has both endpoints contained in a segment in Q or R . Thus if \overline{P}_i has both endpoints contained in a single segment of the subdivided ladder, that segment must be equal to S_j for some index j by construction.

Assume there exists an index j , $1 \leq j \leq l$, and a subset $I' \subseteq I_1$ such that $|I'| \geq (9k^2)^2$ and the path \overline{P}_i has both endpoints in S_j for all $i \in I'$. Let the endpoints of \overline{P}_i be x_i and y_i . By Lemma 4.4, either there exists an edge e on S_j such that at least $9k^2$ distinct \overline{P}_i have one endpoint in each component of $S_j - e$, or there exists $9k^2$ disjoint subpaths of the form $x_i\overline{P}_iy_i$. In the first case, Lemma 4.2 implies G would contain k disjoint non-zero cycles. Thus we may assume that there exist $9k^2$ disjoint subpaths of the form $x_i\overline{P}_iy_i$. We will in fact only need $6k$ such paths to construct k disjoint non-zero cycles. Without loss of generality, we assume that $\overline{P}_1, \overline{P}_2, \dots, \overline{P}_{6k}$ are such that $x_1, y_1, x_2, y_2, \dots, x_{6k}, y_{6k}$ occur on S_j in that order. If $j \neq l$, for all i , $1 \leq i \leq k$, there exists two vertical segments linking S_j to S_{j+1} with endpoints contained in the subpath of S_j linking y_{6i} and $x_{6(i+1)}$. Moreover, these vertical segments can be chosen to be disjoint from the subpath $x_{6i+3}S_jy_{6i+3}$. If $j = l$, we find two such vertical segments linking to S_{l-1} . It follows that G contains k vertex disjoint non-zero theta subgraphs, and consequently, k vertex disjoint non-zero cycles.

We may assume, then, that for every index j , there exist at most $(9k^2)^2$ distinct indices i such that \overline{P}_i has both endpoints contained in S_j . We now show that we may also assume that at most $4k+2$ distinct indices j such that there exists an index i where S_j contains both endpoints of the path \overline{P}_i . Let $J \subseteq \{2, \dots, l-1\}$ be a subset of at least $4k$ indices such that for every $j \in J$, the path S_j contains both endpoints of \overline{P}_i for some index i . Fix $j \in J$ and $i \in I_1$ such that \overline{P}_i has both endpoints in S_j . Let the endpoints of \overline{P}_i be x_i and y_i , and let W be the set of internal vertices of the subpath $x_iS_jy_i$. Given that Γ does not contain an element of order two, either $\overline{P}_i \cup x_iS_jy_i$ forms a non-zero cycle, or one of the paths S_j and $(S_j - W) \cup \overline{P}_i$ forms a non-zero path. Given that $|J| \geq 4k$, we either find k vertex disjoint non-zero cycles, or $3k$ disjoint non-zero paths from Q to R containing k

disjoint non-zero thetas. Either case implies that G contains k disjoint non-zero cycles.

We conclude that there exists a set $I_2 \subseteq I_1$ with $|I_2| \geq |I_1| - (4k + 2)(81k^4) \geq 27816k^6$ such that for every $i \in I_2$ and for every j , $1 \leq j \leq l$, the path \overline{P}_i has at most one endpoint in S_j . By the construction of the \overline{P}_i , it follows that each \overline{P}_i has endpoints in distinct segments of the graph formed by $Q \cup R \cup \left(\bigcup_1^l S_i\right)$. Lemma 4.5 implies that G contains k vertex disjoint non-zero cycles, completing the proof of the lemma. \blacksquare

5 Proof of Theorem 3.1

In this section, we give the proof of Theorem 3.1. We will prove the theorem for the bound $10^{86}k^{87}$ for the value of n_k . These constants are certainly far from optimal. The choice of constants was determined by two goals. First, we wanted to obtain a bound for n_k that is a polynomial in k , and, second, we attempt to make the proof as simple as possible while still keeping such a polynomial bound.

We begin by formalizing what we mean by a minimal counter-example to Theorem 3.1.

Definition Let k be a positive integer. We say the pair $(H, \{P_i : 1 \leq i \leq n\})$ is a k -counter-example if the following conditions hold.

1. The integer n satisfies $n \geq 10^{86}k^{87}$.
2. There exists an integer $l \geq n$ congruent to 0 mod 4 and an integer $m \geq n$ congruent to 7 mod 8, such that H is isomorphic to a subdivision of the $l \times m$ wall.
3. The paths P_1, \dots, P_n are vertex disjoint X -paths where X is the set of branch vertices of H .
4. There exists a group Γ which does not have any elements of order two and a Γ -labeling of $G := H \cup \left(\bigcup_1^n P_i\right)$ such that each P_i is a non-zero path and the graph G does not contain k vertex disjoint non-zero cycles. Such a labeling γ will be referred to as a *testifying* Γ -labeling.

The pair $(H, \{P_i : 1 \leq i \leq n\})$ is k -minimal if it is a k -counter-example and there does not exist a k -counter-example containing fewer edges.

The proof of Theorem 3.1 will proceed by showing that no such k -minimal pair $(H, \{P_i : 1 \leq i \leq n\})$ exists. If the paths P_1, \dots, P_n were disjoint from H except for their endpoints, then the proof of Theorem 3.1 would follow from a relatively straight forward application of Lemma 4.6 and Lemma 4.7. The majority of the proof will be concerned with controlling how the individual P_i intersect with the subgraph H . Towards that end, we give the following definitions. A *leg* of P_i for some i is a subpath of $P_i - E(H)$ forming a $V(H)$ -path. A *foot* of P_i is a component of $P_i \cap H$.

We begin with a series of lemmas.

Lemma 5.1 *Let $(H, \{P_i : 1 \leq i \leq n\})$ form a k -minimal pair. Then for all $1 \leq i \leq n$, no leg of P_i has both endpoints contained in the same segment of H .*

Proof. Let P be a segment of H , and assume there exists a leg Q of some path P_i such that the endpoints x and y of Q are both contained in P . We define the subgraph H' to be equal to $H - \text{int}(xPy) \cup Q$ when xPy has length at least two, and set H' equal to $H - xy \cup Q$ otherwise. In

either case, the subgraph H' is isomorphic to a subdivision of an $l \times m$ wall with the same branch vertices as H . It follows that $(H', \{P_i : 1 \leq i \leq n\})$ contradicts the fact that $(H, \{P_i : 1 \leq i \leq n\})$ is k -minimal. ■

Lemma 5.2 *Let $(H, \{P_i : 1 \leq i \leq n\})$ form a k -minimal pair. Let γ be a testifying Γ -labeling of $H \cup (\bigcup_1^n P_i)$. There exists a subset $I \subseteq \{1, 2, \dots, n\}$ with $|I| \geq n - 10^7 k^8$ such that for every $i \in I$, every leg of P_i has weight zero in the testifying labeling.*

Proof. The claim follows immediately from Lemma 4.7. If there exists a set of at least $10^7 k^8$ distinct indices i such that P_i contains a non-zero leg, then since no leg has both endpoints contained in a segment of H , Lemma 4.7 implies that G contains k vertex disjoint non-zero cycles, a contradiction. ■

For any index i such that the corresponding P_i in a k -minimal pair has only legs of weight zero, there must exist some foot of P_i that has non-zero weight.

Lemma 5.3 *Let $(H, \{P_i : 1 \leq i \leq n\})$ be a k -minimal pair with testifying Γ -labeling γ . Let I be the indices such that P_i has no non-zero leg for all $i \in I$. Let P be a segment of H and let $J \subseteq I$ be the set of indices such that P_i has a non-zero foot on P . Then $|J| \leq 10^{65} k^{77}$.*

Proof. Let $(H, \{P_i : 1 \leq i \leq n\})$, I , Γ and γ be as in the statement. Let $G = H \cup (\bigcup_1^n P_i)$. Fix P to be a segment of H , and let

$$J := \{i \in I : P_i \text{ has a non-zero foot in } P\}.$$

We assume that

$$|J| \geq 10^{65} k^{77} \geq [(4k) [12(10^7(k+1)^8)] (192(9k^2+1)^4) + 4k + 2]^4 10^8 (k+1)^8 + 10^7 (k+1)^8,$$

and we will show that in this case G contains k vertex disjoint non-zero cycles. This contradiction implies the statement of the lemma.

The proof proceeds by repeatedly refining the set J . We will need the following constants.

$$\begin{aligned} n_1 &= [(4k) [12(10^7(k+1)^8)] (192(9k^2+1)^4) + 4k + 2]^4 \\ n_2 &= \sqrt{n_1} \\ n_3 &= \sqrt{n_2} - k - 2 \\ n_4 &= n_3 - 3k \\ n_5 &= n_4 / 192(9k^2+1)^4 \geq (4k) 12(10^7(k+1)^8) \end{aligned}$$

For every $i \in J$, let F_i be a non-zero foot of P_i contained in the segment P . Let the endpoints of F_i be x_i and y_i for all $i \in J$. Let F'_i be the union of F_i and the two legs of P_i intersecting F_i . Observe that for all $i \in J$, the path F'_i is a non-zero path with both endpoints in $V(H) - \text{int}(P)$.

Claim 5.4 *There exists a subset $J_1 \subseteq J$ and a segment Q of H such that for all $i \in J_1$, every path F'_i has both endpoints contained in Q . Moreover, $|J_1| \geq n_1$.*

Proof. Lemma 4.7 implies that the path F'_i has endpoints in distinct segments of H for at most $10^7(k+1)^8$ distinct indices. To see this, assume there exist $10^7(k+1)^8$ such indices. Let e be a new edge with endpoints equal to the endpoints of P . The graph $(H - \text{int}(P)) \cup e \cup \{F'_i : i \in J\}$ would contain $k+1$ non-zero cycles by Lemma 4.7. Consequently, $(H - \text{int}(P)) \cup \{F'_i : i \in J\}$ would contain k disjoint non-zero cycles, a contradiction.

We now fix a subset $J' \subseteq J$ and distinct segments Q_i of H such that for all $i \in J'$, the path F'_i has both endpoints contained in Q_i , and furthermore, for all distinct $i, j \in J'$, $Q_i \neq Q_j$. Assume we can choose J' such that $|J'| \geq 10^8(k+1)^8 \geq 5 \cdot 10^7(k+1)^8$. Observe that for any distinct indices $i, j \in J'$, there exists a non-zero path R with one endpoint in Q_i and the second endpoint in Q_j where $R \subseteq F'_i \cup F'_j \cup x_i P x_j$. As a slight technicality, in order to ensure that the path R has endpoints on distinct segments of H , we require Q_i and Q_j to not have a common endpoint. Since every segment of H has a common endpoint with at most four other segments, we conclude that there exists a subset $J'' \subseteq J'$ with $|J''| \geq |J'|/5$ such that for all $i, j \in J''$, Q_i and Q_j are disjoint. We conclude that there exist $10^7(k+1)^8$ disjoint non-zero paths attaching to $H - \text{int}(P)$. As in the previous paragraph, G contains k disjoint non-zero cycles.

We conclude that there exists a segment Q of H such that at least $n_1 \geq (|J| - 10^7(k+1)^8)/10^8(k+1)^8$ indices i , the path F'_i has both endpoints in the segment Q , and the claim is proven. ■

For the remainder of the proof of Lemma 5.3, we fix J_1 and the segment Q to be as in the statement of Claim 5.4. We fix the labels x'_i and y'_i to be the endpoints of F'_i in Q .

Claim 5.5 *There exists a subset $J_2 \subseteq J_1$ with $|J_2| \geq n_2$ such that for all distinct $i, j \in J_2$, the subpaths $x'_j Q y'_j$ and $x'_i Q y'_i$ are disjoint.*

Proof. We apply Lemma 4.4 to either find an edge e in Q intersecting at least $n_2 \geq 9k^2$ of the paths $x'_i Q y'_i$, or a subset of n_2 distinct indices i such that the paths $x'_i Q y'_i$ are pairwise disjoint. In the first case, there exist $9k^2$ disjoint non-zero paths attaching to the two components of $Q - e$. Lemma 4.2 implies that G would then contain k vertex disjoint non-zero cycles. We conclude that there exists a subset $J_2 \subseteq J_1$ with $|J_2| \geq \sqrt{|J_1|} \geq n_2$ such that the subpaths $x'_i Q y'_i$ are pairwise disjoint for all $i \in J_2$. ■

Let the endpoints of P be u_1 and v_1 and let the endpoints of Q be u_2 and v_2 . There is a natural ordering, denoted by \preceq_P , of the vertices of P where for any two vertices z, z' on P , $z \preceq_P z'$ if z is closer to u_1 on the path P . Similarly, for any two vertices of z and z' on Q , $z \preceq_Q z'$ if z is closer to u_2 on Q . In a slight abuse of notation, we will use \preceq_P to order the feet laying on P where $F_i \preceq_P F_j$ if F_i is closer to the vertex u_1 on the path P . Similarly, we use \preceq_Q to order the subpaths $x'_i Q y'_i$ on Q .

Claim 5.6 *By possibly swapping the labels v_1 and u_1 on P , there exists a set $J_3 \subseteq J_2$ such that $|J_3| = n_3$ and the following hold:*

- i. for all $i, j \in J_3$, if $F_i \preceq_P F_j$, then $x'_i Q y'_i \preceq_Q x'_j Q y'_j$,
- ii. for all $i \in J_3$, the subpath F_i does not intersect the endpoints of P and similarly, the subpath $x'_i Q y'_i$ does not intersect the endpoints of Q , and
- iii. for all $i \in J_3$, $\gamma(x'_i Q y'_i) = -\gamma(F_i)$.

Proof. By the same argument as in the proof of Lemma 4.2, there exists a subset $J' \subseteq J_2$ with $|J'| \geq \lfloor \sqrt{|J_2|} \rfloor$ such that one of the following holds. For all $i, j \in J'$, if $F_i \preceq_P F_j$, then $x'_i Q y'_i \preceq_Q x'_j Q y'_j$, or, for all $i, j \in J'$, if $F_i \preceq_P F_j$, then $x'_j Q y'_j \preceq_Q x'_i Q y'_i$. By possibly swapping the labels u_1 and v_1 on P , we may assume the former holds and that J' satisfies *i*. By possibly discarding the indices corresponding to the first and last feet on P , we may assume that for all $i \in J'$, the subpath F_i does not intersect the endpoints of P and similarly, the subpath $x'_i Q y'_i$ does not intersect the endpoints of Q . Thus J' satisfies *ii*. Finally, observe that if there exist k distinct indices i such that $\gamma(x'_i Q y'_i) \neq -\gamma(F_i)$, then G would contain k disjoint non-zero cycles. By discarding all such indices in J' , we may assume J' satisfies *iii*. as well. We conclude that we have constructed the set J_3 with $|J_3| = \lfloor \sqrt{|J_2|} \rfloor - k - 2 = n_3$, and the claim is proven. \blacksquare

We re-number the paths F_i with $i \in J_3$ so that $J_3 = \{1, 2, \dots, n_3\}$. For all $i \in J_3$, we fix the labels x_i, y_i, x'_i, y'_i of F'_i such that the vertices $x_1, y_1, x_2, y_2, \dots, x_{n_3}, y_{n_3}$ occur on P in that order when traversing from u_1 to v_1 , and and the vertices $x'_1, y'_1, x'_2, y'_2, \dots, x'_{n_3}, y'_{n_3}$ occur on Q in that order when traversing Q from u_2 to v_2 . Throughout the remainder of the proof, as we continue to further refine the set J_3 we will re-number the indices to be the set $\{1, 2, \dots, n'\}$ for some integer n' . When we do so, we do so such that the paths $F_1 \preceq_P F_2 \preceq_P F_3 \preceq_P \dots \preceq_P F_{n'}$.

Observe that for each $i \in J_3$ the two legs contained in F'_i either link x_i to x'_i and link y_i to y'_i , or vice versa and link x_i to y'_i and y_i to x'_i . In the latter case, we say that the path F'_i *twists*.

Claim 5.7 *There are at most $3k$ distinct indices $i \in J_3$ for which the path F'_i twists.*

Proof. Let $i, 2 \leq i \leq |J_3| - 1$, be a fixed index such that the path F'_i twists. There exists two internally disjoint paths linking the endpoints x_i and y_i of the non-zero foot F_i avoiding any internal vertex of F_i . Moreover, these paths can be chosen to use only the vertices of F'_{i-1} and F'_{i+1} as well as the subpaths $x_{i-1} P y_{i+1}$ and $x_{i-1} Q y_{i+1}$. To see this, one such path linking x_i and y_i is formed by the leg $x_i F'_i y'_i$ and a subpath of F'_{i+1} linking $y_i P x_{i+1}$ and $y'_i Q x'_{i+1}$. The second path is obtained from the leg $y_i F'_i x'_i$ and a subpath of F'_{i-1} linking $y'_{i-1} Q x'_i$ and $y_{i-1} P x_i$. It follows that if there existed at least $3k$ distinct indices $i \in J_3$ such that F'_i were twisted, then G would contain k vertex disjoint non-zero thetas, and consequently, k disjoint non-zero cycles. \blacksquare

We conclude, by discarding all indices i such that F'_i twists and by possibly renumbering the remaining feet F_i , that F'_i is not twisted for every $i \in J_4$ where $J_4 = \{1, 2, \dots, n_4\}$.

For each $i \in J_4$, we now want to follow the path P_i starting at each of the vertices x'_i and y'_i to their attachments in $H - (\text{int}(P) \cup \text{int}(Q))$. For all $i \in J_4$, let \overline{F}_i be the subpath of P_i that forms a $V(H - (\text{int}(P) \cup \text{int}(Q)))$ -path containing F_i .

We will use the \overline{F}_i to find many non-zero paths attaching to distinct segments in either $H - \text{int}(Q)$ or in $H - \text{int}(P)$. Applying Lemma 4.7 will then imply that G contains many non-zero cycles. If each \overline{F}_i consisted of F'_i as well as two legs linking x'_i and y'_i to $H - (\text{int}(P) \cup \text{int}(Q))$, then it would be relatively easy to find k disjoint non-zero cycles. The difficulty remaining in the proof of the lemma is that each \overline{F}_i may contain arbitrarily many legs passing between P and Q before leaving $P \cup Q$ to terminate in $H - (\text{int}(P) \cup \text{int}(Q))$.

We define the subpaths D_i and D'_i for $0 \leq i \leq n_4$ as follows. For i such that $1 \leq i \leq n_4 - 1$, let D_i be the path $x_i P x_{i+1} - \{x_{i+1}\}$ and let D'_i be the path $x'_i Q x'_{i+1} - \{x'_{i+1}\}$. We let D_0 be the path $u_1 P x_1 - \{x_1\}$ and let D'_0 be the path $u_2 Q x'_1 - \{x'_1\}$. Let D_{n_4} be the path $x_{n_4} P v_1$ and let D'_{n_4} be the path $x'_{n_4} Q v_2$. A leg S of a path \overline{F}_i for some $i, 1 \leq i \leq n_4$ is *long* if S has one endpoint in D_j and one endpoint in D'_l for indices j and l with $|l - j| \geq 2$.

We now show that there are a bounded number of indices $i \in \{0, 1, \dots, n_4\}$ such that the subpath D_i or D'_i contains an endpoint of a long leg.

Claim 5.8 *There are at most $192(9k^2 + 1)^4$ distinct indices i such that $D_i \cup D'_i$ contains the endpoint of a long leg.*

Proof. To prove this, there are two distinct cases to eliminate: first, when there many distinct pairs (a, b) such that there exists a long leg with one endpoint in D_a and one endpoint in D'_b , and, secondly, when there exists a fixed index a and many distinct indices b such that there exists a long leg with one endpoint in $D_a \cup D'_a$ and one endpoint in $D'_b \cup D_b$.

Case 1: There exist $(9k^2 + 1)^2$ disjoint pairs $\{a, b\}$ such that some long leg has one end in $D_a \cup D'_a$ and one end in $D_b \cup D'_b$. Let $\{\{a(i), b(i)\} : 1 \leq i \leq (9k^2 + 1)^2\}$ be a collection of $(9k^2 + 1)^2$ such disjoint pairs. Assume the pairs are ordered such that $a(i) < b(i)$ for all $1 \leq i \leq (9k^2 + 1)^2$. By applying Lemma 4.3, there exists a subset of $L \subseteq \{1, 2, \dots, (9k^2 + 1)^2\}$ of distinct indices i with $|L| \geq 9k^2 + 1$ such that one of the following holds.

1. There exists an index j such that for every pair $\{a(i), b(i)\}$ for $i \in L$ $a(i) < j$ and $b(i) > j$.
2. Alternatively, for all distinct $i, i' \in L$ either $a(i) \geq b(i')$ or $b(i) \leq a(i')$.

Assume the first possibility holds, and there exists an index j such that for every $i \in L$, $a(i) < j < b(i)$. For all $i \in L$, let R_i be a long leg with endpoints in $D_{a(i)} \cup D'_{a(i)} \cup D_{b(i)} \cup D'_{b(i)}$. As a technicality, we assume that $a(i) \neq 0$ for all $i \in L$. Observe that the union of the subgraph $D_{a(i)} \cup F'_{a(i)} \cup D_{b(i)} \cup F'_{b(i)} \cup R_i$ contains a non-zero path with one end in $D'_{a(i)}$ and one endpoint in $D'_{b(i)}$. It follows that G contains $9k^2$ disjoint non-zero paths with one endpoint in each of the two components of $Q - D'_j$. Lemma 4.2 contradicts our choice of G .

We consider the second possibility when for every distinct $i, i' \in L$ either $a(i) \geq b(i')$ or $b(i) \leq a(i')$. Equivalently, we see that for every distinct $i, i' \in L$, the subpaths of P given by $x_{a(i)}Px_{b(i)+1} - \{x_{b(i)+1}\}$ and $x_{a(i')}Px_{b(i')+1} - \{x_{b(i')+1}\}$ are disjoint. Again, for all $i \in L$, let R_i be a long leg with one endpoint in $D_{a(i)} \cup D'_{a(i)}$ and the other endpoint in $D_{b(i)} \cup D'_{b(i)}$. By the fact that we are considering long leaps, for every $i \in L$, there exists an index $c(i)$ such that $a(i) < c(i) < b(i)$. It now suffices to show that for any index $i \in L$, there exists a non-zero cycle contained in the subgraph $S_i := x_{a(i)}Px_{b(i)+1} - \{x_{b(i)+1}\} \cup x'_{a(i)}Qx'_{b(i)+1} - \{x'_{b(i)+1}\} \cup F'_{a(i)} \cup F'_{b(i)} \cup R_i \cup F'_{c(i)}$. Again, as a technicality, we assume that $a(i) \neq 0$. Then such a non-zero cycle exists since there exists a path linking $x_{c(i)}$ and $y_{c(i)}$ in S_i avoiding $F'_{c(i)} \cup x'_{c(i)}Qy'_{c(i)}$ except at its endpoints by using the path R_i . Consequently, the subgraph S_i contains a non-zero theta, and therefore, a non-zero cycle. Given that $|L| \geq k + 1$, we see that G contains k vertex disjoint non-zero cycles. This contradiction completes the analysis of the first case.

Case 2: There exists an index a and $6[4(9k^2) + 1]^2$ distinct indices b such that some long leg has one end in $D_a \cup D'_a$ and one end in $D_b \cup D'_b$. Fix a to be such an index and define the set $\{\{a, b(i)\} : 1 \leq i \leq 6[4(9k^2) + 1]^2\}$ of pairs be such that there exists a long leg with one endpoint in $D_a \cup D'_a$ and a second endpoint in $D_{b(i)} \cup D'_{b(i)}$. We partition the long legs into those that have an endpoint in D_a and those that have an endpoint in D'_a . We assume that for at least $3[4(9k^2) + 1]^2$ of the pairs $\{a, b(i)\}$ there exists a long leg with one endpoint in D_a and one endpoint in $D'_{b(i)}$. The

analysis would follow similarly when $3[4(9k^2) + 1]^2$ of the pairs $\{a, b(i)\}$ correspond to a long leg with an endpoint in D'_a . For any three such long legs with an endpoint in D_a , we can define a path T with endpoints in D'_j and $D'_{j'}$ for some indices j and j' with $|j - j'| \geq 2$. Thus, we can define paths T_1, T_2, \dots, T_m where for all indices i , $1 \leq i \leq m$, the path T_i has endpoints in D'_j and $D'_{j'}$ for some j and j' with $|j - j'| \geq 2$ where $m \geq [4(9k^2) + 1]^2$. We construct the paths T_1, \dots, T_m to be internally disjoint from $Q \cup (P - D_a) \cup \left(\bigcup_{i \neq a} F'_i\right)$. Notice, for all $1 \leq i \leq m$, the subpath of Q linking the endpoints of T_i contains the path $x'_j Q y'_{j'}$ for some index j . Also, by construction, for all i, j , $1 \leq i, j \leq m$, the path T_i and T_j do not have endpoints in a common $D'_{j'}$ for any index j' . It follows that if there exist $k + 1$ distinct paths T_i such that the subpath of Q defined by the endpoints of T_i are pairwise disjoint, then G would contain k disjoint non-zero thetas, a contradiction. We would need $k + 1$ such non-overlapping paths, as one such path may contain D'_a .

We conclude from Lemma 4.3 that there exists an index a' such that at least $4(9k^2) + 1$ such T_i have an endpoint in each component of $Q - D'_{a'}$. For every $1 \leq i \leq m$, if the path T_i has endpoints in D'_j and $D'_{j'}$, for some indices j and j' , $j, j' \neq a'$, then there exists a non-zero path T'_i with endpoints in P such that $T'_i \subseteq T_i \cup D'_j \cup D'_{j'} \cup D'_{j'+1} \cup F'_j \cup F'_{j'+1}$. To see this, the subgraph $T_i \cup D'_j \cup F'_j \cup D'_{j'}$ contains a path from $P \cap D_j$ to the vertex $x'_{j'+1}$. Then the subgraph $F'_{j'+1} \cup x'_{j'+1} Q y'_{j'+1}$ contains two distinctly weighted paths from $x'_{j'+1}$ to $P \cap D_{j'+1}$ ensuring that the desired non-zero path exists. Each such non-zero T'_i may intersect at most one other T'_i by construction, so we conclude that there are at least $2(9k^2)$ disjoint non-zero paths having one endpoint in each of the two components of $P - e$ for some edge $e \subseteq D_{a'}$. One component of $P - e$ contains the subpath D_a . It follows that there exist two components of $P - \{D_a \cup D_{a'}\}$ and a set of $9k^2$ disjoint non-zero paths with one endpoint in each component. Lemma 4.2 implies G contains k disjoint non-zero cycles, a contradiction.

We have completed the analysis of the two cases. In order to get a general bound on the number of distinct pairs of indices containing a long leg, we consider the following bipartite auxiliary graph. Let the vertex set be $\{u_0, u_1, \dots, u_{n_4}, v_0, v_1, \dots, v_{n_4}\}$ and for all indices i, j , two vertices u_i and v_j adjacent if there exists a long leg with endpoints in D_i and D'_j . From the first case, we see that the auxiliary graph has at most $(9k^2 + 1)^2$ disjoint edges. By König's theorem on matchings in bipartite graphs, we see that the auxiliary graph has subset of the vertices of size at most $(9k^2 + 1)^2$ incident every edge of the auxiliary graph. In the analysis of the second case, we saw that each vertex of the auxiliary graph has degree at most $6[4(9k^2) + 1]^2$. It follows that the auxiliary graph has at most $(9k^2 + 1)^2 6[4(9k^2) + 1]^2 \leq 96(9k^2 + 1)^4$ edges, and, consequently, there are at most $96(9k^2 + 1)^4$ distinct pairs of indices i, j such that there is a long leg with one endpoint in $D_i \cup D'_i$ and one endpoint in $D_j \cup D'_j$. It follows that there are at most $192(9k^2 + 1)^4$ indices i such that $D_i \cup D'_i$ contains an endpoint of a long leg. This completes the proof of the claim. \blacksquare

An immediate consequence of Claim 5.8 is that there exists an index $a \in J_4$ such that for all i , $1 \leq i \leq n_5 = n_4/192(9k^2 + 1)^4$, there does not exist a long leg with an endpoint in $D_{a+i} \cup D'_{a+i}$. We now fix such an index a for the remainder of the proof of Lemma 5.3. For any index i , $1 \leq i \leq n_5$, we say that $D_{a+i} \cup D'_{a+i}$ is *criss-crossed* if there exist distinct indices $j, j' \in \{1, 2, \dots, n_4\} - \{a + i\}$, and subpaths E_j and $E_{j'}$ of $\overline{F_j}$ and $\overline{F_{j'}}$, respectively, such that the following holds:

1. E_j and $E_{j'}$ each have one endpoint in $D_{a+i+1} \cup D'_{a+i+1}$ and one endpoint in $D_{a+i-1} \cup D'_{a+i-1}$,
2. every leg of E_j and $E_{j'}$ has both endpoints in the set $D_{a+i} \cup D'_{a+i}$, $D_{a+i-1} \cup D'_{a+i-1}$, and $D_{a+i+1} \cup D'_{a+i+1}$, and

3. there do not exist subpaths of E_j and $E_{j'}$ satisfying 1. and 2.

We now see that there are at most a bounded number of indices i where $D_{a+i} \cup D'_{a+i}$ is criss-crossed.

Claim 5.9 *There are at most $4k$ distinct indices i , $1 \leq i \leq n_5$ such that $D_{a+i} \cup D'_{a+i}$ is criss-crossed.*

Proof. Let i be an index such that $D_{a+i} \cup D'_{a+i}$ is criss-crossed. Let j and j' be indices and E_j and $E_{j'}$ be the two paths as in the definition of criss-crossed. We define the subgraph H' to be the subgraph formed by $x_{a+i-1}Px_{a+i+2} \cup x'_{a+i-1}Qx'_{a+i+2} \cup F'_{a+i-1} \cup F'_{a+i} \cup F'_{a+i+1} \cup F'_{a+i+2} \cup E_j \cup E_{j'}$. To prove the claim, it suffices to show that H' contains a non-zero cycle. We do this by showing that H' contains a non-zero theta. Define the cycles C and C' such that $C = x_{a+i-1}F'_{a+i-1}x'_{a+i-1} \cup x'_{a+i-1}Qx'_{a+i} \cup x'_{a+i}F'_{a+i}x_{a+i} \cup x_{a+i}Px_{a+i-1}$ and $C' = y_{a+i}F'_{a+i}y'_{a+i} \cup y'_{a+i}Qx'_{a+i+2} \cup x'_{a+i+2}F'_{a+i+2}x_{a+i+2} \cup x_{a+i+2}Py_{a+i}$. Observe that $D_{a+i-1} \cup D'_{a+i-1} \subseteq V(C)$ and $D_{a+i+1} \cup D'_{a+i+1} \subseteq V(C')$. Consequently, there exists two vertex disjoint subpaths of E_j and $E_{j'}$ each with one endpoint in C and one endpoint in C' . It follows that H' contains two internally disjoint paths linking x_{a+i} and y_{a+i} avoiding the non-zero path F_{a+i} . We conclude that H' contains a non-zero theta, and the claim is proven. ■

Given that $n_5 \geq (4k)12(10^7(k+1)^8)$, an immediate consequence of Claim 5.9 is that there exists an index $b \in \{a+1, a+2, \dots, a+n_5\}$ such that for all $1 \leq i \leq 12(10^7(k+1)^8)$, $D_{b+i} \cup D'_{b+i}$ is not criss-crossed. We are now able to complete the lemma. For i , $4 \leq i \leq 12(10^7(k+1)^8) - 3$, one of the paths \bar{F}_{b+i-1} , \bar{F}_{b+i} , or \bar{F}_{b+i+1} must contain a leg with one endpoint in the set $\bigcup_{i-2 \leq i' \leq i+2} (D_{b+i'} \cup D'_{b+i'})$ and one endpoint in $H - (P \cup Q)$. This follows from our choice of b to avoid indices that are criss-crossed and long leaps. If none of the paths \bar{F}_{b+i-1} , \bar{F}_{b+i} , nor \bar{F}_{b+i+1} has such a leg, then each must intersect either the set $D_{b+i-3} \cup D'_{b+i-3}$ or the set $D_{b+i+3} \cup D'_{b+i+3}$ since there is no long leg attaching to any of the sets $D_{b+i-2} \cup D'_{b+i-2}, \dots, D_{b+i+2} \cup D'_{b+i+2}$. It follows that either the set $D_{b+i+2} \cup D'_{b+i+2}$ or the set $D_{b+i-2} \cup D'_{b+i-2}$ is criss-crossed, a contradiction.

We conclude that there exist $2(10^7(k+1)^8)$ distinct legs $L_1, L_2, \dots, L_{2(10^7(k+1)^8)}$ with one endpoint in $P \cup Q$ and one endpoint in $H - (P \cup Q)$. Moreover, if we let the endpoint of L_i in $P \cup Q$ be contained in $D_{\pi(i)} \cup D'_{\pi(i)}$ for some $\pi(i)$, then for all distinct i and i' , the sets $D_{\pi(i)} \cup D'_{\pi(i)}$ and $D_{\pi(i')} \cup D'_{\pi(i')}$ are disjoint by construction. For each i , $1 \leq i \leq 2(10^7(k+1)^8)$, consider the subgraph $L_i \cup D_{\pi(i)} \cup D'_{\pi(i)} \cup F'_{\pi(i)}$. If L_i has one endpoint in P , then the subgraph $L_i \cup D_{\pi(i)} \cup D'_{\pi(i)} \cup F'_{\pi(i)}$ contains a non-zero path with one endpoint in Q and one endpoint in $H - (P \cup Q)$ and otherwise disjoint from Q . Symmetrically, if L_i has one endpoint in Q , there exists a non-zero path with one endpoint in P and one endpoint in $H - (P \cup Q)$ and otherwise disjoint from P . We conclude, without loss of generality, that there exist $10^7(k+1)^8$ disjoint non-zero paths, each with one endpoint in P and one endpoint in $H - (P \cup Q)$ and otherwise disjoint from $H - (P \cup Q)$. Lemma 4.7 implies that G contains k disjoint non-zero cycles, a contradiction. This completes the proof of Lemma 5.3. ■

One final step remains before we give the proof of Theorem 3.1. Consider a k -minimal pair $(H, \{P_1, \dots, P_n\})$. Let $G = H \cup (\bigcup_1^n P_i)$, and let γ be a testifying Γ -labeling of G . Lemma 5.3 implies that there exist many disjoint segments containing a non-zero foot of some path P_i . For each such index, we will define a “breaking path”. The final step of the proof of Theorem 3.1 will be to show that many breaking paths imply the existence of k disjoint non-zero cycles.

A segment S of H with endpoints x and y is *substantially split* by a vertex z , $z \neq x, y$, if the paths xSz and ySz have distinct weights, i.e. $\gamma(xSz) \neq \gamma(ySz)$. A *breaking path* is a $V(H)$ -path Q

with endpoints in distinct segments S_1 and S_2 of H such that at least one of the segments S_1 or S_2 is substantially split by an endpoint of Q . A *splitting vertex* of P'_i is an endpoint P'_i that substantially splits a segment of H .

Consider a segment S of H containing a non-zero foot of some path P_i . If $\gamma(S) = 0$, then one of the legs attaching to such a foot must be a breaking path. Thus Lemma 5.3 implies that a k -minimal pair must contain many breaking paths such that their respective splitting vertices will lay on distinct segments of H . The final lemmas in this section will show how to convert such breaking paths into many disjoint non-zero cycles.

Lemma 5.10 *Let H be isomorphic to a subdivision of the $l \times m$ grid with $l \geq 8$ congruent to $0 \pmod 4$ and $m \geq 23$ congruent to $7 \pmod 8$. Let $\{P_i : i \in I\}$ be a collection of disjoint $V(H)$ -paths such that no P_i has endpoints contained in a single segment of H . Let $G = H \cup (\bigcup_i P_i)$ and let γ be a Γ -labeling of G by a group Γ which does not contain any elements of order two. Assume $\gamma(P_i) = 0$ for all $i \in I$. For all $i \in I$, let z_i be an endpoint of P_i such that*

1. z_i is a splitting vertex of some segment of H , and
2. z_i and z_j are not contained in the same segment for all distinct $i, j \in I$.

If $|I| \geq 10^{20}k^{10}$, then G contains k vertex disjoint non-zero cycles.

Proof. Let $G, H, I, \{P_i : i \in I\}, \{z_i : i \in I\}, \Gamma$ and γ be as in the statement. Let

$$|I| \geq 10^{20}k^{10} \geq 16(20)^2 \cdot 4 \left[((16 \cdot 5^2 27k^2)(16 \cdot 15^2)(10^7 k^8)) + 24k^2 \right] + 16(265)^2 k$$

For all $i \in I$, let w_i be the endpoint of P_i not equal to z_i . Let $\{v_{i,j} : 1 \leq i \leq l, 1 \leq j \leq m\}$ be vertices of H corresponding to the vertices of the wall. Specifically, if X is the branch vertices of H , then X has the canonical labeling. We let $Z = \bigcup_{i \in I} z_i$. Assume, to reach a contradiction, that G does not contain k disjoint non-zero cycles.

First, we observe that many of the paths P_i have endpoints that are “distant” in the subgraph H . We define the function $\lambda(x, y)$ for any two vertices $x, y \in V(H)$ to be the the minimum number of branch vertices in X contained in a subpath of H linking x and y . If $Y \subseteq V(H)$ is a subset of vertices, then $\lambda(x, Y)$ is $\min_{y \in Y} \lambda(x, y)$. Note that the function λ satisfies the triangle inequality. For all integers $t \geq 5$ and vertices $v \in V(H)$, the set of vertices x with $\lambda(x, v) \leq t$ is contained in a subgraph of H isomorphic to a subdivision of the $(2t + 3) \times (4t + 1)$ wall. It follows that there exist at most $(2t + 3)(4t) + (2t + 2)(2t + 1) \leq 12t^2 + 18t + 3 \leq 16t^2$ distinct segments containing a vertex x with $\lambda(x, v) \leq t$.

We claim that there exists a set $I_1 \subseteq I$ such that

1. for all $i \in I_1$, $\lambda(z_i, w_i) > 32$, and,
2. for all $i, j \in I_1$, $i \neq j$, $\lambda(z_i, z_j) > 20$.

Furthermore, the set I_1 may be chosen so that $|I_1| \geq (|I| - 16(265)^2 k) / (16 \cdot 20^2)$. To see this, let $I' \subseteq I$ be a set of indices such that $\lambda(z_i, w_i) \leq 32$ for all $i \in I'$. Assume, to reach a contradiction, that $|I'| \geq 16(265)^2 k$.

Let $i \in I'$ and let S be the segment of H such that $z_i \in V(S)$. Let the endpoints of S be $v_{a,b}$ and $v_{a',b'}$. Let X_i be defined to be $\{v_{i,j} : |a - i| \leq 66, |b - j| \leq 33\}$. This set contains every branch vertex $v_{i,j}$ with $\lambda(v_{i,j}, v_{a,b}) \leq 33$. Note that we must consider indices at distance 66 to cover the case when

$a = 1$ and the branch vertices are of the form $v_{1,2j+1}$ for $j \geq 0$. Let W_i be the subgraph consisting of every segment of W with both endpoints in X_i . Then W_i is a subdivision of a wall of height at most 67 and width at most 122. Moreover, by construction, both w_i and z_i are contained in W_i . The subgraph $W_i \cup P_i$ must then contain a non-zero theta, and consequently, a non-zero cycle.

We now show that the subgraphs W_i are pair-wise disjoint for at least k of the indices of I' . Observe that for every vertex of $x \in V(W_i)$, $\lambda(x, z_i) \leq 66 + 66 = 132$. It follows that for all $i, j \in I'$, if z_i and z_j satisfy $\lambda(z_i, z_j) \geq 265$, then W_i and W_j are disjoint. There are less than $16(265)^2$ distinct segments S such that $x \in V(S)$ satisfies $\lambda(x, z_i) \leq 265$. It follows that for at least $|I'| / (16(265)^2) \geq k$ distinct indices in I' , the subgraphs W_i are pairwise disjoint. This implies that G contains k disjoint non-zero cycles, a contradiction.

We conclude that no such set I' of indices exists. It follows that for at least $|I| - 16(265)^2 k$ indices i in I , $\lambda(z_i, w_i) > 32$. Moreover, for every index $i \in I$, there exist at most $16(20)^2$ distinct indices j such that $\lambda(z_j, z_i) \leq 20$. It follows that the desired set I_1 exists with $|I_1| \geq (|I| - 16(265)^2 k) / (16 \cdot 20^2)$.

We now define a subgraph H' of H isomorphic to a subdivision of a smaller wall such that H' does not contain many of the breaking vertices z_i . In a slight abuse of notation, if x and y are two vertices of H , we use xHy to refer to the unique path P of H linking x and y where every internal vertex of P has degree two in H , whenever such a unique path exists. Let S_i , $1 \leq i \leq l$, be the path $\bigcup_{j=1}^{m-1} v_{i,j} H v_{i,j+1}$. We define R_i , $1 \leq i \leq (m-3)/2$, to be the path $\bigcup_{j=1}^{l/2} v_{2j-1, 2i-1} H v_{2j, 2i-1} \cup \bigcup_{j=1}^{l/2-1} v_{2j, 2i} H v_{2j+1, 2i} \cup \bigcup_{j=2^{l-1}} v_{j, 2i-1} H v_{j, 2i}$.

Intuitively speaking, we will select half of the paths S_i to form the horizontal paths of our new wall, and half of the paths R_i to form the vertical paths of our new wall. Moreover, we will do so to ensure that at least a quarter of all the vertices of Z are not contained in the new wall. Towards that end, for each pair of horizontal paths S_i , we keep the one with fewer vertices z_j . Explicitly, we define the set J_1 of indices as follows. For every i , $1 \leq i \leq l/2$, if $|V(S_{2i-1}) \cap Z| \leq |V(S_{2i}) \cap Z|$, then $2i-1 \in J_1$. Otherwise $2i \in J_1$. If we let $Z' = \{z \in Z : z \notin V(S_j) \quad \forall j \in J_1\}$, we see that $|Z'| \geq |Z|/2$. Similarly, we define J_2 as follows. For every i , $1 \leq (m-3)/4$, if $|V(R_{2i-1}) \cap Z'| \leq |V(R_{2i}) \cap Z'|$, then $2i-1 \in J_2$. Otherwise, $2i \in J_2$. The set $\bigcup_{i \in J_1} S_i \cup \bigcup_{i \in J_2} R_i$ contains a subgraph H' isomorphic to a $l/2 \times (m-3)/4$ wall. Observe that every segment of H' consists of at most 17 distinct segments of H . The remainder of the proof will show that we have many non-zero paths attaching to distinct segments of the subgraph H' .

We begin with several easy observations concerning the subgraph H' . By the construction of H' , there exists a set $I_2 \subseteq I_1$ with $|I_2| \geq |I_1|/4$ such that $z_i \notin V(H')$ for all $i \in I_2$. Also, observe that for every index $i \in J_1$, the path $v_{i,4} S_i v_{i,m-3}$ is contained in the subgraph H' . It follows that for every index $i \in I_2$, $\lambda(z_i, V(H')) \leq 5$.

For every $i \in I_2$, we define two paths T_i^1 and T_i^2 as follows. Let T_i^1 be a path in H linking z_i and $V(H')$ intersecting as few branch vertices of H as possible. If the segment of H containing Z_i is Q , let C be a cycle of H containing Q and at most six branch vertices of H . The cycle C corresponds to a facial cycle in the natural embedding of the $l \times m$ wall in the plane. Then let T_i^2 be a path linking z_i and $V(H')$ in $T_i^1 \triangle C$ the symmetric difference of T_i^1 and C . Note that T_i^1 contains at most five branch vertices of H , and T_i^2 contains at most 10 such branch vertices. Also, if Q has endpoints equal to u and v , then T_i^1 contains exactly one of the two subpaths $z_i Q u$ and $z_i Q v$, say $z_i Q u$, and T_i^2 then contains the subpath $z_i Q v$. By our choice of z_i to be a splitting vertex, then either $\gamma(T_i^1) \neq \gamma(T_i^2)$ and one of the paths T_i^1 or T_i^2 is a non-zero path, or one of the T_i^1 and T_i^2 contains a non-zero segment of H . Also, for all $i, j \in I_2$, $i \neq j$, it follows that $T_i^1 \cap T_j^1 = \emptyset$, $T_i^1 \cap T_j^2 = \emptyset$ by the fact that $\lambda(z_i, z_j) > 20$.

Lemma 4.6 implies that H has at most $24k^2$ distinct segments of non-zero weight. It follows that there exists a subset $I_3 \subseteq I_2$ with $|I_3| \geq |I_2| - 24k^2$ such that for all $i \in I_3$, $\gamma(T_i^1) \neq \gamma(T_i^2)$.

Assume that there exists a segment Q of H that is internally disjoint from $V(H')$ and a subset $I'' \subseteq I_3$ with $|I''| \geq 27k^2$ such that $w_i \in \text{int}(Q)$ for all $i \in I''$. Then either T_i^1 or T_i^2 is a non-zero path with exactly one endpoint in $\text{int}(Q)$ and one endpoint in H' and otherwise is disjoint from the two. Since there exists three paths, not necessarily disjoint, in H' covering every vertex of H' , we see that there exists a path Q' contained in H' and $9k^2$ distinct indices i such that T_i^1 is a non-zero path with one endpoint in Q and one endpoint in Q' . Lemma 4.2 implies that G contains k disjoint non-zero cycles, a contradiction.

Recall that for any $i \in I_3$ there are at most $16 \cdot t^2$ segments containing vertices x with $\lambda(x, w_i) < t$. It follows that for any $i \in I_3$, there are at most $16 \cdot 15^2$ distinct indices j with $\lambda(w_i, z_j) \leq 15$. Also, by the above argument, there are at most $(16 \cdot 5^2) 27k^2$ distinct indices j where w_j is contained in a segment Q' of H not contained in H' with the property $\lambda(w_i, w_j) \leq 5$. we conclude that there exists a subset $I_4 \subseteq I_3$ with $|I_4| \geq |I_3| / ((16 \cdot 5^2 27k^2)(16 \cdot 15^2))$ such that if $i \in I_4$, then for all $j \in I_4$,

1. $\lambda(w_i, z_j) > 15$, and
2. if w_j is contained in a segment Q' not contained in H' , then $\lambda(w_i, w_j) > 5$.

For every $i \in I_4$, let T'_i be a path linking w_i and $V(H')$ in H containing a minimal number of branch vertices of H . Then by our construction of I_4 , the either $T'_i \cup P_i \cup T_i^1$ or $T'_i \cup P_i \cup T_i^2$ is non-zero path with both endpoints in H' . Moreover, it is a non-zero path with endpoints in distinct segments of H' . This follows from the fact that every segment of H' contains at most 17 segments of H , the paths T_i^1 and T_i^2 each contain at most 10 branch vertices of H , the path T'_i contains at most five branch vertices of H , and yet $\lambda(w_i, z_i) > 32$. Given that $|I_4| \geq 10^7 k^8$, we conclude from Lemma 4.7 that G contains k disjoint non-zero cycles. This final contradiction completes the proof of the lemma. \blacksquare

We now give the proof of Theorem 3.1.

Proof. [Theorem 3.1]

We prove Theorem 3.1 with the constant $c = 10^{86}$. Let $k \geq 1$ be given and let

$$n = 10^{86} k^{87} \geq (2(10^{20} k^{10}) + 24k^2) 10^{65} k^{77} + 10^7 k^8.$$

We assume the theorem is false. It suffices to consider a minimal k -counter-example $(H, \{P_i : 1 \leq i \leq n\})$ with testifying Γ -labeling γ , and we will show that such a graph indeed does contain k disjoint non-zero cycles. This contradiction would then imply the theorem. Let l and m be as in the definition of k -counter-example. Let $G = H \cup (\bigcup_1^n P_i)$.

We first observe that there exists a set $I \subseteq \{1, 2, \dots, n\}$ with $|I| \geq n - 10^7 k^8$ such that for all $i \in I$, the path P_i does not contain a non-zero leg. This follows from Lemma 5.2. For every $i \in I$, the path P_i must have a non-zero foot contained in some segment of H . Lemma 5.3 implies that for every segment S of H , the path P_i has a non-zero foot in S for at most $10^{65} k^{77}$ distinct indices $i \in I$. Thus there exists an integer m , $m \geq |I| / (10^{65} k^{77}) = 2(10^{20} k^{10})$ and segments S_i , $1 \leq i \leq m$ such that for all indices i there exists some index $j \in I$ such that S_i contains a non-zero foot of the path P_j .

We consider the segments S_i for $1 \leq i \leq m$. For every i such that $\gamma(S_i) = 0$, it follows that there exists a breaking path Q_i attaching to a splitting vertex on the segment S_i . By construction, for all

$1 \leq i \leq j \leq m$, the path Q_i is disjoint from Q_j unless $Q_i = Q_j$ and each endpoint of Q_i is a breaking vertex on S_i and S_j , respectively. We conclude that there exists $10^{20}k^{10}$ disjoint breaking paths attaching to H . Lemma 5.10 implies that G contains k disjoint non-zero cycles. This contradicts our choice of G to be a minimal k -counter-example, completing the proof of Theorem 3.1. ■

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