Relation between pairs of representations of signed binary matroids

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Abstract

We show how pairs of signed graphs with the same even cycles relate to pairs of grafts with the same even cuts. These results are proved in the more general context of signed binary matroids.

1 Introduction

We assume that the reader is familiar with the basics of matroid theory. See Oxley [4] for the definition of the terms used here. We will only consider binary matroids in this paper. Thus the reader should substitute the term "binary matroid" every time "matroid" appears in this text.

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Given a graph G and $X \subseteq E(G)$, $G[X]$ denotes the graph induced by X. A subset C of edges is a *cycle* if $G[C]$ is a graph where every vertex has even degree. An inclusion-wise minimal non-empty cycle is a *circuit*. We denote by $\mathrm{cycle}(G)$ the set of all cycles of G. Since the cycles of G correspond to the cycles of the *graphic matroid* represented by G, we identify $\mathrm{cycle}(G)$ with that matroid. Given two graphs G_1 and G_2 such that $\mathrm{cycle}(G_1) = \mathrm{cycle}(G_2)$, what is the relation between G_1 and G_2? This is answered by the following theorem of Whitney [8].

Theorem 1. *For a pair of graphs G_1 and G_2,* $\mathrm{cycle}(G_1) = \mathrm{cycle}(G_2)$ *if and only if G_1 and G_2 are related by a sequence of Whitney-flips.*

We need to define the term "Whitney-flip". Consider a graph G and a partition $X, \bar{X}$ of $E(G)$ where $|X|, |\bar{X}| \geq 2$ and $V(G[X]) \cap V(G[\bar{X}]) = \{u_1, u_2\}$, for some $u_1, u_2 \in V(G)$. Let G' be obtained by identifying vertices u_1, u_2 of $G[X]$ with vertices u_2, u_1 of $G[\bar{X}]$ respectively. Then G and G' are related by a *Whitney-flip*. A Whitney-flip will also be the operation which consists of identifying two vertices from distinct components, as well as the operation consisting of partitioning the graph into components each of which is a block of G. An example of two graphs related by a Whitney-flip is given in Figure 1. In this example the set X is given by edges $5, 6, 9, 10$. As Whitney-flips preserve cycles, sufficiency of the previous theorem is immediate.

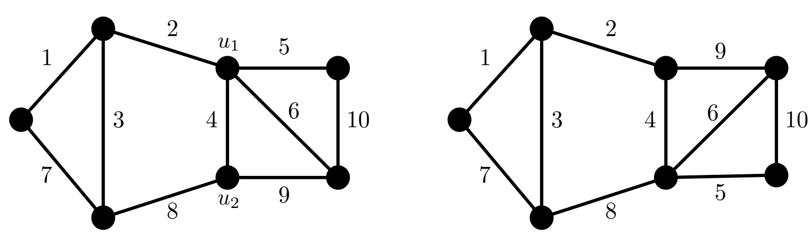

Figure 1: Example of a Whitney-flip.

Given a graph G, we denote by $\mathrm{cut}(G)$ the set of all cuts of G. Since the cuts of G correspond to the cycles of the *cographic matroid* represented by G, we identify $\mathrm{cut}(G)$ with that matroid. Since the matroids $\mathrm{cut}(G)$ and $\mathrm{cycle}(G)$ are duals of one another, we can reformulate Theorem 1 as follows.

Theorem 2. *For a pair of graphs G_1 and G_2, $\mathrm{cut}(G_1) = \mathrm{cut}(G_2)$ if and only if G_1 and G_2 are related by a sequence of Whitney-flips.*

A *signed graph* is a pair (G,Σ) where $\Sigma \subseteq E(G)$. We call Σ a *signature* of G. A subset $D \subseteq E(G)$ is Σ-*even* (resp. Σ-*odd*) if $|D \cap \Sigma|$ is even (resp. odd). When there is no ambiguity we omit the prefix Σ when referring to Σ-even and Σ-odd sets. In particular we refer to odd and even edges and cycles. We denote by $\mathrm{ecycle}(G,\Sigma)$ the set of all even cycles of (G,Σ). It can be verified that $\mathrm{ecycle}(G,\Sigma)$ is the set of cycles of a binary matroid (with ground set $E(G)$) which we call the *even cycle matroid* represented by (G,Σ). We identify $\mathrm{ecycle}(G,\Sigma)$ with that matroid. Signed graphs are a special case of biased graphs [9] and even cycle matroids are a special case of lift matroids [10].

A *graft* is a pair (G,T) where G is a graph, $T \subseteq V(G)$ and $|T|$ is even. A cut $\delta_G(U) := \{uv \in E(G) : u \in U, v \notin U\}$ is T-*even* (resp. T-*odd*) if $|U \cap T|$ is even (resp. odd). When there is no ambiguity we omit the prefix T when referring to T-even and T-odd cuts. We denote by $\mathrm{ecut}(G,T)$ the set of all even cuts of (G,T). It can be verified that $\mathrm{ecut}(G,T)$ is the set of cycles of a binary matroid (with ground set $E(G)$) which we call the *even cut matroid* represented by (G,T). We identify $\mathrm{ecut}(G,T)$ with that matroid.

Question 1: Let (G_1,Σ_1) and (G_2,Σ_2) be signed graphs.

When is the relation $\mathrm{ecycle}(G_1,\Sigma_1) = \mathrm{ecycle}(G_2,\Sigma_2)$ satisfied?

Question 2: Let (G_1,T_1) and (G_2,T_2) be grafts.

When is the relation $\mathrm{ecut}(G_1, T_1) = \mathrm{ecut}(G_2, T_2)$ satisfied?

Theorem 1 anwers Question 1 for the case where $\Sigma_1 = \Sigma_2 = \emptyset$ and Theorem 2 answers Question 2 for the case where $T_1 = T_2 = \emptyset$. Slilaty [6] addresses a special case of the corresponding problem for biased matroids. The corresponding problem for bicircular matroids has been widely studied and a complete characterization of when two graphs represent the same bicircular matroid is known (see [7], [1] and [3]).

We know that Theorems 1 and 2 are equivalent. In an upcoming paper we shall provide a complete answer to Question 1 for three general classes of signed graphs.

The main result of this paper is Theorem 4, which illustrates the relation between Question 1 and Question 2. We introduce and explain this result in Section 2. In Section 3 we generalize to matroids the concepts introduced in the previous section and we state and prove more general results about signed binary matroids. Section 3.2 shows how the results on signed matroids imply the results for signed graphs and grafts; this section also provides a more detailed version of Theorem 4 (in Proposition 17). Section 3.3 introduces a matroid construction that leads to matroid-preserving operations on graphs.

2 Even cycles and even cuts

2.1 Matrix representations

Even cycle and even cut matroids are binary matroids: we now explain how to obtain their matrix representation from a signed graph or a graft representation.

Let (G, Σ) be a signed graph. Let $A(G)$ be the incidence matrix of G, i.e. the columns of

$A(G)$ are indexed by the edges of G, the rows of $A(G)$ are indexed by the vertices of G and entry (v,e) of $A(G)$ is 1 if vertex v is incident to edge e in G and 0 otherwise. Then $A(G)$ is a matrix representation of the graphic matroid represented by G. We obtain a matrix representation of ecycle(G,Σ) as follows. Let S be the transpose of the characteristic vector of Σ; hence S is a row vector indexed by $E(G)$ and S_e is 1 if $e \in \Sigma$ and 0 otherwise. Let A be the binary matrix obtained from $A(G)$ by adding row S. Let $M(A)$ be the binary matroid represented by A. Every cycle C of $M(A)$ corresponds to a cycle of G; moreover, C intersects Σ with even parity. Thus $M(A) = \text{ecycle}(G,\Sigma)$. Suppose we add an odd loop Ω to (G,Σ). This corresponds to adding a new column to A which has all zero entries, except for row S, which contains a one. Hence contracting Ω in the new matrix we obtain the matrix $A(G)$. It follows that every even cycle matroid is a lift of a graphic matroid. Note that in constructing A we may replace $A(G)$ with any binary matrix whose rows span the cut space of G.

Now we describe how to obtain a matrix representation of an even cut matroid from a graft representation. For a graph G, a set $J \subseteq E(G)$ is a *T-join* if T is the set of vertices of odd degree of $G[J]$. Let (G,T) be a graft and J a T-join of G. Let $\hat{A}(G)$ be a binary matrix whose rows span the cycle space of G. Hence $\hat{A}(G)$ is a matrix representation of the cographic matroid represented by G. Let $\hat{S}$ be the transpose of the incidence vector of J, that is, $\hat{S}$ is a row vector indexed by $E(G)$ and $\hat{S}_e$ is 1 if $e \in J$ and 0 otherwise. Construct a matrix $\hat{A}$ from $\hat{A}(G)$ by adding row $\hat{S}$. Let $M(\hat{A})$ be the binary matroid represented by $\hat{A}$. Let C be a cycle of $M(\hat{A})$; then C corresponds to a cut of G and intersects J with even parity. It is easy to see that a cut intersects J with even parity if and only if it is T-even. Thus $M(\hat{A}) = \text{ecut}(G,T)$. Suppose we uncontract an odd bridge Ω in (G,Σ). This corresponds to adding a new column to $\hat{A}$ which has all zero entries, except for row S, which contains a one; this is because Ω is not contained in any cycle of G and Ω is contained in every T-join of G. Hence contracting Ω in the new matrix we obtain the matrix $\hat{A}(G)$. It follows that every even cut matroid is a lift of a cographic matroid.

2.2 Isomorphism

The following theorem of Gerards, Lovász, Schrijver, Seymour, Shih and Truemper (see [2]) answers Question 1 for the class of even cycle matroids within the class of graphic matroids.

Theorem 3. *Let (G,Σ) and (G',Σ') be signed graphs. Suppose that* $\mathrm{ecycle}(G,\Sigma) = \mathrm{ecycle}(G',\Sigma')$ *and that this matroid is graphic. Then (G,Σ) and (G',Σ') are related by a sequence of Whitney-flips, signature exchanges, and Lovász-flips.*

We need to define the terms "signature exchange" and "Lovász-flip". A set $\Sigma' \subseteq E(G)$ is a *signature* of (G,Σ) if $\mathrm{ecycle}(G,\Sigma) = \mathrm{ecycle}(G,\Sigma')$. It can be readily checked that Σ' is a signature of (G,Σ) if and only if $\Sigma' = \Sigma \triangle D$ for some cut D of G. The operation that consists of replacing a signature of a signed graph by another signature is a *signature exchange*.

Given a graph G we denote with $\mathrm{loop}(G)$ the set of loops of G. Consider a signed graph (G,Σ) and vertices $v_1, v_2 \in V(G)$ where $\Sigma \subseteq \delta_G(v_1) \cup \delta_G(v_2) \cup \mathrm{loop}(G)$. We can construct a signed graph (G',Σ) from (G,Σ) by replacing the adjacencies of every odd edge e as follows:

- if $e = v_1v_2$ in G, then e becomes a loop in G' incident to v_1;
- if e is a loop in G, then $e = v_1v_2$ in G';
- if $e = xv_i$, for $1 \leq i \leq 2$ and $x \neq v_1, v_2$, then $e = xv_{3-i}$ in G'.

In this case, we say that (G',Σ) is obtained from (G,Σ) by a *Lovász-flip*. An example of two signed graphs related by a Lovász-flip is given in Figure 2, where the white vertices represent the vertices v_1, v_2 in the definition. Note that Lovász-flips are only well defined on signed graphs (G,Σ) where $\Sigma \subseteq \delta_G(v_1) \cup \delta_G(v_2) \cup \text{loop}(G)$ for some vertices v_1 and v_2 . It can be easily verified that Lovász-flips preserve even cycles.

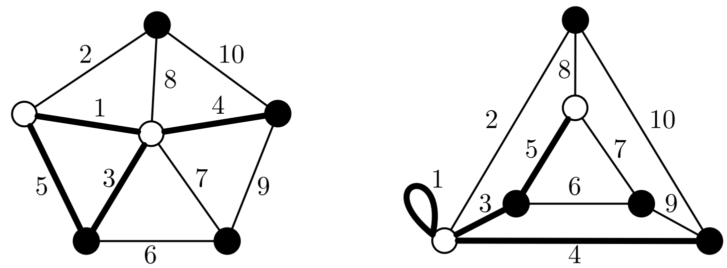

Figure 2: Signed graphs related by a Lovász-flip. Bold edges are odd.

An interesting result of Shih (see [5]) characterizes the relation between graphs G and H such that the cycle space of H is a codimension-1 subspace of the cycle space of G. In particular, this characterizes when $\text{cycle}(H) = \text{ecycle}(G,\Sigma)$ for a graph H and a signed graph (G,Σ). In fact, if $\text{cycle}(H)$ and $\text{cycle}(G)$ have the same dimension, then by Theorem 1 they are related by Whitney-flips. If they have different dimensions, then $\text{cycle}(H)$ is a codimension-1 subspace of $\text{cycle}(G)$, because, for a fixed odd cycle C of (G,Σ), every other odd cycle D of G can be written as the symmetric difference of C and the even cycle $C \triangle D$.

2.3 Equivalence of Question 1 and 2

Suppose that $\text{cycle}(G_1) = \text{cycle}(G_2)$ for graphs G_1 and G_2. Then $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$ if and only if $\Sigma_1 \triangle \Sigma_2$ is a cut of G_1 (equivalently of G_2). Moreover, $\text{ecut}(G_1,T_1) = \text{ecut}(G_2,T_2)$ if and only if any T_1-join of G_1 is a T_2-join of G_2 (recall that, for a graph H, a set $J \subseteq E(H)$ is a *T-join* if T is the set of vertices of odd degree of $H[J]$). Thus Questions 1 and 2 are trivial for the case when $\mathrm{cycle}(G_1) = \mathrm{cycle}(G_2)$. A signed graph (G, Σ) is *bipartite* if G has no Σ-odd cycle. The following theorem shows the relation between the two questions in the case $\mathrm{cycle}(G_1) \neq \mathrm{cycle}(G_2)$.

Theorem 4. *Let G_1 and G_2 be graphs such that* $\mathrm{cycle}(G_1) \neq \mathrm{cycle}(G_2)$.

(1) Suppose there exists a pair $\Sigma_1, \Sigma_2 \subseteq E(G_1)$ such that $\mathrm{ecycle}(G_1, \Sigma_1) = \mathrm{ecycle}(G_2, \Sigma_2)$. *For $i = 1, 2$, if (G_i, Σ_i) is bipartite define $C_i := \emptyset$, otherwise let C_i be a Σ_i-odd cycle of G_i. Let T_{3-i} be the vertices of odd degree in $G_{3-i}[C_i]$. Then* $\mathrm{ecut}(G_1, T_1) = \mathrm{ecut}(G_2, T_2)$.

(2) Suppose there exists a pair $T_1 \subseteq V(G_1)$ and $T_2 \subseteq V(G_2)$ (where $|T_1|$ and $|T_2|$ are even) such that $\mathrm{ecut}(G_1, T_1) = \mathrm{ecut}(G_2, T_2)$. *For $i = 1, 2$, if $T_i = \emptyset$ let $\Sigma_{3-i} = \emptyset$, otherwise let $t_i \in T_i$ and let $\Sigma_{3-i} := \delta_{G_i}(t_i)$. Then* $\mathrm{ecycle}(G_1, \Sigma_1) = \mathrm{ecycle}(G_2, \Sigma_2)$.

The proof of Theorem 4 is postponed until Section 3.2. We illustrate this result with an example. Consider the signed graphs (G_i, Σ_i), for $i = 1, 2, 3$, in Figure 3. (G_2, Σ_2) is obtained from (G_1, Σ_1) by a Lovász-flip on vertices b, f; (G_3, Σ_3) is obtained from (G_2, Σ_2) by first a signature exchange $\Sigma_3 := \Sigma_2 \triangle \delta_{G_2}(b)$, then by performing a Lovász-flip on vertices a, f. As Lovász-flips and signature exchanges preserve even cycles, $\mathrm{ecycle}(G_1, \Sigma_1) = \mathrm{ecycle}(G_3, \Sigma_3)$. In the same

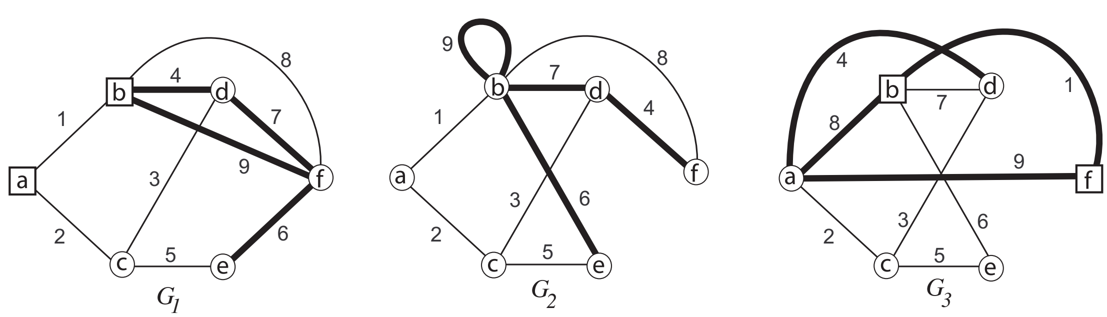

Figure 3: Bold edges of G_i are in Σ_i, square vertices of G_1, G_3 are T_1, T_3.

figure consider the grafts (G_1,T_1) and (G_3,T_3) where $T_1=\{a,b\}$ and $T_3=\{b,f\}$. These grafts are obtained using the construction in Theorem 4(1). If we consider the odd cycle $\{4,7,9\}$ of (G_1,Σ_1), then T_3 is the set of vertices of odd degree in $G_3[\{4,7,9\}]$. Similarly, if we consider the odd cycle $\{1,8,9\}$ of (G_3,Σ_3), then T_1 is the set of vertices of odd degree in $G_1[\{1,8,9\}]$. The theorem states that $\mathrm{ecut}(G_1,T_1)=\mathrm{ecut}(G_3,T_3)$. We can also consider the reverse construction as in Theorem 4(2). Pick $a\in T_1$, then $\delta_{G_1}(a)=\{1,2\}$. Now $\{1,2\}\triangle\Sigma_3$ is a cut of G_3, hence $\{1,2\}$ is a signature of (G_3,Σ_3). Similarly, pick $b\in T_3$, then $\delta_{G_3}(b)=\{1,6,7,8\}$ is a signature of (G_1,Σ_1).

The following results will follow immediately from Proposition 11 and Remark 16.

Remark 5. *Given graphs G_1 and G_2 with* $\mathrm{cycle}(G_1)\neq\mathrm{cycle}(G_2)$*, there exists, up to signature exchanges, at most one pair Σ_1,Σ_2 such that* $\mathrm{ecycle}(G_1,\Sigma_1)=\mathrm{ecycle}(G_2,\Sigma_2)$ *and there exists at most one pair T_1,T_2 such that* $\mathrm{ecut}(G_1,T_1)=\mathrm{ecut}(G_2,T_2)$.

The condition that $\mathrm{cycle}(G_1)\neq\mathrm{cycle}(G_2)$ is necessary for uniqueness, as otherwise any pair $\Sigma_1=\Sigma_2$ will yield the same even cycles.

Suppose we can answer Question 1, does Theorem 4 then provide us with an answer to Question 2? Consider grafts (G,T) and (G',T'), where $\mathrm{ecut}(G,T)=\mathrm{ecut}(G',T')$. By Theorem 4 there exist signatures Σ and Σ' such that $\mathrm{ecycle}(G,\Sigma)=\mathrm{ecycle}(G',\Sigma')$. Suppose we know a sequence of signed graphs (G_i,Σ_i), for $i\in[n]$, with $(G,\Sigma)=(G_1,\Sigma_1)$ and $(G',\Sigma')=(G_n,\Sigma_n)$, where (G_i,Σ_i) have the same even cycles for all $i\in[n]$. (For example, this is the case when $T'=\emptyset$ (i.e. $\mathrm{ecut}(G',T')$ is cographic) as then $\Sigma=\emptyset$ (i.e. $\mathrm{ecycle}(G,\Sigma)$ is graphic) and Theorem 3 then describes how (G,Σ) and (G',Σ') are related.) Can we find sets T_i, for all $i\in[n]$, such that $T=T_1,T'=T_n$ where (G_i,T_i) have the same even cuts for all $i\in[n]$? The example in Figure 3 shows that this is not always the case. Remark 5 states that graphs G_1 and G_3 determine T_1 and T_3 uniquely. However, it is not possible to find a set T_2 such that $\mathrm{ecut}(G_1,T_1) = \mathrm{ecut}(G_2,T_2)$, because the edge 9 is a loop in G_2 but is contained in the T_1-even cut $\{6,7,8,9\}$ of G_1.

This leads to the following definition: a set of graphs $\{G_1,\dots,G_n\}$ is *harmonious* if, for all distinct $i,j \in [n]$, $\mathrm{cycle}(G_i) \neq \mathrm{cycle}(G_j)$ and there exist $\Sigma_1,\dots,\Sigma_n$ and $T_1,\dots,T_n$ such that $\mathrm{ecycle}(G_i,\Sigma_i) = \mathrm{ecycle}(G_j,\Sigma_j)$ and $\mathrm{ecut}(G_i,T_i) = \mathrm{ecut}(G_j,T_j)$, for all $i,j \in [n]$. For instance, the set $\{G_1,G_2,G_3\}$ in Figure 3 is not harmonious. In fact no large set of graphs is harmonious.

Theorem 6. *Suppose that $\{G_1,\dots,G_n\}$ is a harmonious set of graphs. Then $n \leq 3$.*

Note that, in contrast, it is easy to construct arbitrarily large sets of graphs $\{G_1,\dots,G_n\}$ such that, for all distinct $i,j \in [n]$, $\mathrm{cycle}(G_i) \neq \mathrm{cycle}(G_j)$ and for which there exist $\Sigma_1,\dots,\Sigma_n$ such that $\mathrm{ecycle}(G_i,\Sigma_i) = \mathrm{ecycle}(G_j,\Sigma_j)$, for all $i,j \in [n]$.

The bound of 3 is best possible. A construction that yields a harmonious set of 3 graphs $\{G_1,G_2,G_3\}$ is as follows: let (G_1,Σ_1) be any signed graph with vertices u,v where $\Sigma_1 \subseteq \delta_{G_1}(u) \cup \delta_{G_1}(v)$. Let (G_2,Σ_2) be obtained from (G_1,Σ_1) by a Lovász-flip on u,v, and let (G_3,Σ_3) be obtained from $(G_1,\Sigma_1 \triangle \delta_{G_1}(u))$ by a Lovász-flip on u,v. Finally, let $T_1 = \{u,v\}$ and for $i = 2,3$, let T_i be the vertices in G_i corresponding to u,v.

The proof of Theorem 6 follows immediately from the proof of Theorem 15 in Section 3.1.4.

3 Generalization to signed matroids

In this section we will generalize to matroids the concepts introduced in the previous section. Recall that all the matroids considered in this work are binary. Let M be a matroid and let $\Sigma \subseteq E(M)$. A *cycle* of M is any subset $C \subseteq E(M)$ such that C is equal to the union of pairwise disjoint circuits. A *cocycle* is a cycle of the dual M^* of M. A pair (M,Σ), where $\Sigma \subseteq E(M)$, is a *signed matroid*. A subset $D \subseteq E(M)$ is Σ-*even* (resp. Σ-*odd*) if $|D \cap \Sigma|$ is even (resp. odd). The set of all cycles of M that are Σ-even forms the set of cycles of a matroid which we denote by $\operatorname{ecycle}(M,\Sigma)$. We say that Σ' is a *signature* of (M,Σ) if $\operatorname{ecycle}(M,\Sigma) = \operatorname{ecycle}(M,\Sigma')$. It can be readily checked that Σ' is a signature of (M,Σ) if and only if $\Sigma' = \Sigma \triangle D$ for some cocycle D of M. The operation that consists of replacing a signature of a signed matroid by another signature is called *signature exchange*. A signed matroid (M,Σ) is *bipartite* if all cycles of M are Σ-even. It can be easily verified that (M,Σ) is bipartite if and only if Σ is a cocycle of M. When $M = \operatorname{cycle}(G)$ for some graph G, then $\operatorname{ecycle}(M,\Sigma) = \operatorname{ecycle}(G,\Sigma)$ and the reader should verify that in this context the aforementioned definitions for signed matroids correspond to the definitions for signed graphs.

3.1 Results for signed matroids

3.1.1 Pairs

Let (M_1,Σ_1) and (M_2,Σ_2) be signed matroids such that $\operatorname{ecycle}(M_1,\Sigma_1) = \operatorname{ecycle}(M_2,\Sigma_2)$. A cycle (resp. cocycle) C of M_1 is *preserved* from M_1 to M_2 if C is also a cycle (resp. cocycle) of M_2. A signature of (M_1,Σ_1) is *preserved* if it is a signature of (M_2,Σ_2). Isomorphism problems for signed binary matroids arise in pairs.

Theorem 7. *Let (M_1,Σ_1) and (M_2,Σ_2) be signed matroids with $\operatorname{ecycle}(M_1,\Sigma_1) = \operatorname{ecycle}(M_2,\Sigma_2)$. Then there exist $\Gamma_1,\Gamma_2 \subseteq E(M_1)$ such that $\operatorname{ecycle}(M_1^*,\Gamma_1) = \operatorname{ecycle}(M_2^*,\Gamma_2)$ and, for $i = 1,2$, the Γ_i-even cocycles of M_i are exactly the preserved cocycles of M_i. Moreover, if (M_i,Σ_i) is bipartite, then so is (M_{3-i}^*,Γ_{3-i}).*

The proof requires a number of preliminaries and is postponed to the end of the section. Given a signed matroid (M,Σ), the cocycles of $\text{ecycle}(M,\Sigma)$ are the sets that intersect every Σ-even cycle of M with even cardinality. Thus we have the following.

Remark 8. *The cocycles of* $\text{ecycle}(M,\Sigma)$ *are the cocycles of M and the signatures of* (M,Σ)*,*

which in turns implies the following.

Remark 9. *Suppose that* $\text{ecycle}(M_1,\Sigma_1) = \text{ecycle}(M_2,\Sigma_2)$.

(1) If D is a non-preserved cocycle of M_1 then D is a signature of (M_2,Σ_2).

(2) If D is a non-preserved signature of (M_1,Σ_1) *then D is a cocycle of M_2.*

Proof. For both (1) and (2), Remark 8 implies that D is a cocycle of $\text{ecycle}(M_1,\Sigma_1)$ and hence of $\text{ecycle}(M_2,\Sigma_2)$. Remark 8 implies that D is either a cocycle of M_2 or a signature of (M_2,Σ_2). For (1) D is not a cocycle of M_2, for (2) D is not a signature of (M_2,Σ_2). □

Lemma 10. *Suppose that* $\text{ecycle}(M_1,\Sigma_1) = \text{ecycle}(M_2,\Sigma_2)$*, for signed matroids* (M_1,Σ_1) *and* (M_2,Σ_2)*. For $i=1,2$, there exists $\Gamma_i \subseteq E(M_i)$ such that, for every cocycle D of M_i, D is preserved if and only if it is Γ_i-even. Moreover, if (M_{3-i},Σ_{3-i}) is bipartite, then $\Gamma_i = \emptyset$.*

Proof. Fix $i \in \{1,2\}$. Let B be a cobasis of M_i. For any $e \notin B$, let D_e denote the unique cocircuit in $B \cup \{e\}$ (these are the fundamental cocircuits of M_i). Then we let $e \in \Gamma_i$ if and only if D_e is non-preserved. Consider now an arbitrary cocycle D of M_i. D may be expressed as the symmetric difference of a set of distinct fundamental cocircuits D_e, where, say, s of these are non-preserved. By construction, $|D \cap \Gamma_i| = s$. By Remark 9(1), non-preserved cocycles of M_i are signatures of (M_{3-i},Σ_{3-i}). Moreover, the symmetric difference of an even (resp. odd) number of signatures of (M_{3-i},Σ_{3-i}) is a cocycle of M_{3-i} (resp. a signature of (M_{3-i},Σ_{3-i})). It follows that D is a cocycle of M_{3-i} when s is even and is a signature of (M_{3-i},Σ_{3-i}) when s is odd. If (M_{3-i},Σ_{3-i}) is non-bipartite, then signatures of (M_{3-i},Σ_{3-i}) are not cocycles of M_{3-i} and the result follows. If (M_{3-i},Σ_{3-i}) is bipartite, then every cocycle of M_i is preserved. As a consequence, $\Gamma_i=\emptyset$ and the result follows as well. □

Proof of Theorem 7. Lemma 10 implies that, for $i=1,2$, there exists $\Gamma_i\subseteq E(M_1)$ such that the preserved cocycles of M_i are exactly the Γ_i-even cocycles of M_i. Hence, $\mathrm{ecycle}(M_1^*,\Gamma_1)=\mathrm{ecycle}(M_2^*,\Gamma_2)$. Again by Lemma 10, if (M_i,Σ_i) is bipartite, then $\Gamma_{3-i}=\emptyset$, so (M_{3-i}^*,Γ_{3-i}) is bipartite. □

3.1.2 Uniqueness

The main observation in this section is the following.

Proposition 11. *Suppose (M_1,Σ_1) and (M_2,Σ_2) are signed matroids such that $M_1\neq M_2$ and* $\mathrm{ecycle}(M_1,\Sigma_1)=\mathrm{ecycle}(M_2,\Sigma_2)$. *For $i=1,2$, the Σ_i-even cycles of M_i are exactly the preserved cycles of M_i. In particular, Σ_1 and Σ_2 are unique up to signature exchanges.*

Proposition 11 follows directly from the next remark.

Remark 12. *Suppose* $\mathrm{ecycle}(M_1,\Sigma_1)=\mathrm{ecycle}(M_2,\Sigma_2)$. *If C is a Σ_1-odd cycle of M_1 which is preserved, then* $\mathrm{cycle}(M_1)=\mathrm{cycle}(M_2)$.

Proof. For any odd cycle D of (M_1,Σ_1), let $B:=D\triangle C$. As B is an even cycle of (M_1,Σ_1), we know that B is an even cycle of (M_2,Σ_2), hence D is an odd cycle of (M_2,Σ_2). Hence, $\mathrm{cycle}(M_1)\subseteq\mathrm{cycle}(M_2)$. As C is a cycle of M_1 and M_2 and C is Σ_1-odd, C is also a preserved Σ_2-odd cycle of M_2. Hence by symmetry the reverse inclusion holds as well. □

3.1.3 Odd cycles and signatures

Remark 13. *Suppose that* $\mathrm{ecycle}(M_1,\Sigma_1) = \mathrm{ecycle}(M_2,\Sigma_2)$*, for signed matroids* (M_1,Σ_1) *and* (M_2,Σ_2)*, where* $M_1 \neq M_2$*. If* (M_1,Σ_1) *is bipartite, then let* $\Sigma := \emptyset$*. Otherwise there exists a non-preserved cocycle* D *of* M_2*; let* $\Sigma := D$*. Then* Σ *is a signature of* (M_1,Σ_1)*.*

Proof. We may assume that (M_1,Σ_1) is non-bipartite. By Theorem 7, there exist Γ_1 and Γ_2 such that, for $i = 1,2$, the Γ_i-even cocycles of M_i are exactly the preserved cocycles of M_i. If every cocycle of M_2 is preserved, then (M_2^*,Γ_2) is bipartite. It follows, from Theorem 7 applied to (M_1^*,Γ_1) and (M_2^*,Γ_2), and from Proposition 11, that (M_1,Σ_1) is bipartite, a contradiction. Hence, some cocycle D of M_2 is non-preserved. The result then follows by Remark 9(1). $\square$

The signature Σ of (M_1,Σ_1) in Remark 13 is called an M_2-*standard signature*.

Theorem 14. *Let* (M_1,Σ_1) *and* (M_2,Σ_2) *be signed matroids such that* $M_1 \neq M_2$ *and let* $\Gamma_1 \subseteq E(M_1)$ *and* $\Gamma_2 \subseteq E(M_2)$*. Assume that* $\mathrm{ecycle}(M_1,\Sigma_1) = \mathrm{ecycle}(M_2,\Sigma_2)$ *and* $\mathrm{ecycle}(M_1^*,\Gamma_1) = \mathrm{ecycle}(M_2^*,\Gamma_2)$*. If, for* $i = 1,2$*,* Σ_i *is an* M_{3-i}*-standard signature, then for any* $D \subseteq E(M_1)$ *the following hold.*

(1) Suppose that (M_1,Σ_1) *is non-bipartite. Then*

D is a Σ_1-odd cycle of M_1 if and only if D is a Σ_2-even signature of (M_2^*,Γ_2).

(2) Suppose that (M_1,Σ_1) *and* (M_2,Σ_2) *are non-bipartite. Then*

D is a Σ_1-odd signature of (M_1^*,Γ_1) if and only if D is a Σ_2-odd signature of (M_2^*,Γ_2).

Proof. We begin with the proof of (1). Let D be a Σ_1-odd cycle of M_1. Remark 12 implies that D is non-preserved. Remark 9(1) implies that D is a signature of (M_2^*,Γ_2). If $\Sigma_2 = \emptyset$, then D is trivially Σ_2-even. Otherwise, as Σ_2 is a standard signature, Σ_2 is a cocycle of M_1. Since M_1 is a binary matroid, cycles and cocycles have an even intersection, hence D is Σ_2-even. Conversely, let D be a Σ_2-even signature of (M_2^*, Γ_2). As (M_1, Σ_1) is non-bipartite, there exists a Σ_1-odd cycle C of M_1. By the first part of the proof, C is a Σ_2-even signature of (M_2^*, Γ_2). Therefore $C \triangle D$ is a Σ_2-even cycle of M_2, hence a Σ_1-even cycle of M_1. Thus D is a Σ_1-odd cycle of M_1.

We now proceed with the proof of (2). Let D be a Σ_1-odd signature of (M_1^*, Γ_1). Moreover, let C be a Σ_1-odd cycle of M_1. Then $D \triangle C$ is a Σ_1-even signature of (M_1^*, Γ_1). By part (1) and symmetry between M_1 and M_2, $D \triangle C$ is a Σ_2-odd cycle of M_2. Also, by part (1), C is a Σ_2-even signature of (M_2^*, Γ_2). Hence $D = (D \triangle C) \triangle C$ is a Σ_2-odd signature of (M_2^*, Γ_2). Hence every Σ_1-odd signature of (M_1^*, Γ_1) is a Σ_2-odd signature of (M_2^*, Γ_2). The other inclusion follows by symmetry between M_1 and M_2. □

3.1.4 Harmonious sets

A set of matroids $\{M_1, \ldots, M_n\}$ is *harmonious* if $M_i \neq M_j$, for all distinct $i, j \in [n]$, and there exist signatures $\Sigma_1, \ldots, \Sigma_n$ and $\Gamma_1, \ldots, \Gamma_n$ such that $\text{ecycle}(M_i, \Sigma_i) = \text{ecycle}(M_j, \Sigma_j)$ and $\text{ecycle}(M_i^*, \Gamma_i)$ $= \text{ecycle}(M_j^*, \Gamma_j)$, for all $i, j \in [n]$.

Theorem 15. *Suppose that $\{M_1, \ldots, M_n\}$ is a harmonious set of matroids. Then $n \leq 3$.*

Proof. Suppose for a contradiction that there exists a harmonious set $\{M_1, \ldots, M_4\}$. Note that, by Proposition 11, $\Sigma_1, \ldots, \Sigma_4$, $\Gamma_1, \ldots, \Gamma_4$ are unique up to signature exchange. First suppose that (M_k, Σ_k) is bipartite for some $k \in [4]$. Then, by Theorem 7, (M_i^*, Γ_i) is bipartite for every $i \in [4] \setminus \{k\}$. Hence, for $i, j \in [4] \setminus \{k\}$, the matroids M_i and M_j have the same cocycles, hence $M_i = M_j$, a contradiction. Therefore, for every $i \in [4]$, (M_i, Σ_i) is non-bipartite and by duality (M_i^*, Γ_i) is non-bipartite as well. By Theorem 7, a cocycle C of M_4 is non-preserved if and only if it is Γ_4-odd. We fix C to be an odd cocycle of (M_4,Γ_4), and conclude that C is non-preserved for M_i, for all $i \in [3]$. By definition, C is an M_4-standard signature for (M_i,Σ_i), for all $i \in [3]$.

For every $i \in [3]$, let C_i be a C-odd signature of (M_i^*,Γ_i). Note that such signatures exist because (M_i,Σ_i) is non-bipartite, hence the symmetric difference of an odd circuit of (M_i,Σ_i) and the signature of (M_i^*,Γ_i) will be an odd signature. By Theorem 14(2), C_i is a signature of (M_4^*,Γ_4), for every $i \in [3]$. The symmetric difference of two signatures of (M_4^*,Γ_4) is a cycle of M_4. Moreover, for some $j,k \in [3]$ with $j \neq k$, C_j and C_k have the same parity with respect to Σ_4. Hence $D := C_j \triangle C_k$ is a Σ_4-even cycle of M_4, so D is a Σ_i-even cycle of M_i for every $i \in [4]$. Therefore $C_j = D \triangle C_k$ is a C-odd signature of both (M_j^*,Γ_j) and (M_k^*,Γ_k). Now let C' be a Σ_4-odd cycle of M_4. By Theorem 14(1), C' is a C-even signature of (M_j^*,Γ_j) and (M_k^*,Γ_k). Therefore $C_j \triangle C'$ is a C-odd cycle of both M_j and M_k. Hence, by Remark 12, $M_j = M_k$, a contradiction. $\square$

3.2 Applications to signed graphs and grafts

In this section we show how the results for signed matroids apply to signed graphs and grafts.

Remark 16. *Let (G,T) be a graft, let Γ be a T-join of G and let $M = cut(G)$.*

(1) A cut of G is T-even if and only if it is Γ-even. In particular, $\mathrm{ecut}(G,T) = \mathrm{ecycle}(M,\Gamma)$.

(2) A set of edges is a T-join of G if and only if it is a signature of (M,Γ).

Proof of Theorem 4. We begin with the proof of (1). We omit the cases when (G_1,Σ_1) or (G_2,Σ_2) is bipartite. For $i = 1,2$, let $M_i := \mathrm{cycle}(G_i)$. By Theorem 7, there exist Γ_1 and Γ_2 such that $\mathrm{ecycle}(M_1^*,\Gamma_1) = \mathrm{ecycle}(M_2^*,\Gamma_2)$. Since C_i is an odd cycle of (M_i,Σ_i), it is non-preserved.

It follows from Remark 9(1) that C_i is a signature of (M^*_{3-i},Γ_{3-i}). Hence, $\text{ecycle}(M^*_1,C_2) = \text{ecycle}(M^*_2,C_1)$. Let T_i be the vertices of odd degree in $G_i[C_{3-i}]$. Remark 16(1) implies that $\text{ecut}(G_1,T_1) = \text{ecut}(G_2,T_2)$.

We proceed with the proof of (2). We omit the cases when $T_1 = \emptyset$ or $T_2 = \emptyset$. For $i = 1,2$, let $M_i := \text{cut}(G_i)$ and let Γ_i be a T_i-join of G_i. Remark 16(1) implies that $\text{ecycle}(M_1,\Gamma_1) = \text{ecycle}(M_2,\Gamma_2)$. By Theorem 7 there exist $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ such that $\text{ecycle}(M^*_1,\tilde{\Sigma}_1) = \text{ecycle}(M^*_2,\tilde{\Sigma}_2)$. As $\Sigma_i = \delta_{G_{3-i}}(t_{3-i})$ is a T_{3-i}-odd cut of G_{3-i}, by Remark 16(1) Σ_i is a Γ_{3-i}-odd cycle of (M_{3-i},Γ_{3-i}). It follows from Remark 9(1) that Σ_i is a signature of $(M^*_i,\tilde{\Sigma}_i)$. We conclude that $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(M^*_1,\Sigma_1) = \text{ecycle}(M^*_2,\Sigma_2) = \text{ecycle}(G_2,\Sigma_2)$. □

Suppose that $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$ and $\text{ecut}(G_1,T_1) = \text{ecut}(G_2,T_2)$, where $\text{cycle}(G_1) \neq \text{cycle}(G_2)$. If (G_1,Σ_1) is bipartite, let $\Sigma := \emptyset$. Otherwise, by Remark 13, there exists a T_2-odd cut D of (G_2,T_2); let $\Sigma := D$. Then Σ is a standard signature of (G_1,Σ_1). Given a signature $\tilde{\Sigma}_i$ of (G_i,Σ_i), $\Sigma_i \triangle \tilde{\Sigma}_i$ is a cut D of G_i. We say that $\tilde{\Sigma}_i$ is *T_i-even* (resp. *T_i-odd*) if D is a T_i-even (resp. T_i-odd) cut.

Proposition 17. *Suppose that* $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$ *and* $\text{ecut}(G_1,T_1) = \text{ecut}(G_2,T_2)$, *where* $\text{cycle}(G_1) \neq \text{cycle}(G_2)$. *If Σ_1 and Σ_2 are standard signatures, the following hold.*

(1) Suppose that (G_1,Σ_1) is non-bipartite. Then

D is a Σ_1-odd cycle of G_1 if and only if D is a Σ_2-even T_2-join of G_2.

(2) Suppose that (G_1,Σ_1) and (G_2,Σ_2) are non-bipartite. Then

D is a Σ_1-odd T_1-join of G_1 if and only if D is a Σ_2-odd T_2-join of G_2.

(3) Suppose that $T_1 \neq \emptyset$. Then

D is a T_1-odd cut of G_1 if and only if D is T_2-even signature of (G_2,Σ_2).

(4) Suppose that $T_1, T_2 \neq \emptyset$. Then

D is a T_1-odd signature of (G_1, Σ_1) if and only if D is T_2-odd signature of (G_2, Σ_2).

We illustrate Proposition 17 on the example in Figure 3. We have that $\Sigma_1' := \delta_{G_3}(f) = \{1,9\}$ is a standard signature of (G_1, Σ_1) and $\Sigma_3' := \delta_{G_1}(a) = \{1,2\}$ is a standard signature of (G_3, Σ_3). Then the odd cycle $\{4,7,9\}$ of (G_1, Σ_1') is a Σ_3'-even T_3-join of G_3. The set $\{1\}$ is a Σ_1'-odd T_1-join of G_1 and a Σ_3'-odd T_3-join of G_3. Moreover $\{1,3,5\} = \delta_{G_1}(\{a,c\})$ is a T_1-odd cut of G_1. As $\{1,3,5\} \triangle \Sigma_3' = \{2,3,5\} = \delta_{G_3}(c)$, $\{1,3,5\}$ is a T_3-even signature of (G_3, Σ_3'). Finally, $\{2,9\}$ is a T_1-odd signature of (G_1, Σ_1') which is also a T_3-odd signature of (G_3, Σ_3').

Proof of Proposition 17. We prove parts (1) and (3) only, as statements (2) and (4) follow similarly from Theorem 14(2). We begin with the proof of (1). For $i = 1,2$, let $M_i := \text{cycle}(G_i)$. Clearly, D is a cycle of G_1 if and only if D is a cycle of M_1. Let Γ_2 be a T_2-join of G_2. Remark 16(2) implies that D is a T_2-join of G_2 if and only if D is a signature of (M_2^*, Γ_2). The result now follows from Theorem 14(1). We proceed with the proof of (3). For $i = 1,2$, let $M_i := \text{cut}(G_i)$ and let Γ_i be a T_i-join of G_i. Remark 16(1) implies that D is a T_1-odd cut of G_1 if and only if D is a Γ_1-odd cycle of M_1. Since Σ_2 is a standard signature of (M_2^*, Σ_2), Σ_2 is a Γ_1-odd cycle of M_1. It follows from Theorem 14(1) that Σ_2 is Γ_2-even. D is a T_2-even signature of (G_2, Σ_2) if and only if D is a signature of (M_2^*, Σ_2) where $\Sigma_2 \triangle D$ is T_2-even. Equivalently, by Remark 16(1), $\Sigma_2 \triangle D$ is Γ_2-even. As Σ_2 is Γ_2-even, this occurs if and only if D is Γ_2-even. The result now follows from Theorem 14(1). $\square$

3.3 Whitney-flips and Lovász-flips - a unified view

In this section we explain how Whitney-flips and Lovász-flips arise from a simple matroid construction. We will apply the same construction to even cut matroids to derive an operation on grafts that preserves even cuts. Given a matroid M and disjoint subsets I and J of $E(M)$, we denote by $M/I \setminus J$ the matroid obtained from M by contracting I and deleting J. We require the following observation.

Lemma 18. *Let M be a matroid and let a,b,c,d denote distinct elements of M. Suppose that $\{a,b,c,d\}$ is both a cycle and a cocycle of M. Then $M/\{a,b\} \setminus \{c,d\} = M \setminus \{a,b\}/\{c,d\}$.*

Proof. Let $M_1 := M/\{a,b\} \setminus \{c,d\}$ and let $M_2 := M \setminus \{a,b\}/\{c,d\}$. We want to show that the cycles of M_1 are exactly the cycles of M_2. By symmetry between M_1 and M_2 (and between $\{a,b\}$ and $\{c,d\}$), it suffices to show that every cycle of M_1 is a cycle of M_2. Let C be any cycle of M_1 . Then there exists a cycle D of M such that $C \subseteq D \subseteq C \cup \{a,b\}$. Since $\{a,b,c,d\}$ is a cocycle of M and M is binary, $|D \cap \{a,b,c,d\}|$ is even. Hence, either none of a,b are in D or both of a,b are in D. In the former case, $D = C$ and C is cycle of M_2 as required. In the latter case, $D = C \cup \{a,b\}$. Since $\{a,b,c,d\}$ is a cycle of M, $D \triangle \{a,b,c,d\} = C \cup \{c,d\}$ is a cycle of M. It follows that C is cycle of M_2. □

Given a graph G and disjoint subsets I and J of $E(G)$, we denote by $G/I \setminus J$ the graph obtained from G by contracting the edges in I and deleting the edges in J. We need to define minors for signed graphs and grafts. Let (G,Σ) be a signed graph and let $e \in E(G)$. Then $(G,\Sigma) \setminus e$ is defined as $(G \setminus e, \Sigma \setminus \{e\})$. $(G,\Sigma)/e$ is equal to $(G \setminus e, \emptyset)$ if e is an odd loop of (G,Σ); to $(G \setminus e, \Sigma)$ if e is an even loop of (G,Σ); otherwise $(G,\Sigma)/e$ is equal to $(G/e,\Gamma)$, where Γ is any signature of (G,Σ) which does not contain e. Let (G,T) be a graft and let $e \in E(G)$ with endpoints s and

t. Then $(G,T)/e$ is defined as $(G/e,T')$, where $T' := T \setminus \{s,t\}$ if both of s,t or none of s,t are in T, and $T' := T \setminus \{s,t\} \cup \{u\}$, where u is the vertex corresponding to the edge e in G/e, if exactly one of s and t are in T. $(G,T) \setminus e$ is equal to $(G \setminus e, \emptyset)$ if e is an odd cut of (G,T); otherwise $(G,T) \setminus e$ is equal to $(G \setminus e, T)$. The following are easy to verify:

$$\begin{aligned} \text{cycle}(G)/I \setminus J &= \text{cycle}\left(G/I \setminus J\right) \\ \text{ecycle}(G,\Sigma)/I \setminus J &= \text{ecycle}\left((G,\Sigma)/I \setminus J\right) \\ \text{ecut}(G,T)/I \setminus J &= \text{ecut}\left((G,T) \setminus I/J\right) \end{aligned} \tag{1}$$

3.3.1 Whitney-flips

Consider a graph G which consists of components $G[X_1]$ and $G[X_2]$, for some partition X_1, X_2 of $E(G)$. For $i = 1,2$, pick vertices s_i and t_i in $G[X_i]$. Denote by C the set of edges $\{a,b,c,d\}$, where $a = (s_1,t_1), b = (s_2,t_2), c = (s_1,t_2)$ and $d = (s_2,t_1)$. Let H be the graph obtained from G by adding the edges in C. Since C is a circuit and a cut of H, it is a cycle and a cocycle of cycle(H). Lemma 18 implies that cycle$(H) \setminus \{a,b\}/\{c,d\} = $ cycle$(H)/\{a,b\} \setminus \{c,d\}$. It follows from (1) that cycle$(H \setminus \{a,b\}/\{c,d\}) = $ cycle$(H/\{a,b\} \setminus \{c,d\})$. It can now be easily verified that $H \setminus \{a,b\}/\{c,d\}$ and $H/\{a,b\} \setminus \{c,d\}$ are related by a Whitney-flip and that any two graphs related by a single Whitney-flip can be obtained in that way. In particular, graphs related by Whitney-flips have the same set of cycles.

3.3.2 Lovász-flips

Consider a graph G. Pick vertices s_1,t_1,s_2 and t_2 of G. Denote by C the set of edges $\{a,b,c,d\}$, where $a = (s_1,t_1), b = (s_2,t_2), c = (s_1,t_2)$ and $d = (s_2,t_1)$. Let H be the graph obtained from

G by adding the edges in C. Since C is an even cycle of (H,C), it is a cycle of ecycle(H,C). Since C is a signature of (H,C), it is a cocycle of ecycle(H,C) (see Remark 8). Lemma 18 implies that ecycle$(H,C) \setminus \{a,b\}/\{c,d\} =$ ecycle$(H,C)/\{a,b\} \setminus \{c,d\}$. It follows from (1) that ecycle$\big((H,C) \setminus \{a,b\}/\{c,d\}\big) =$ ecycle$\big((H,C)/\{a,b\} \setminus \{c,d\}\big)$. It can now be easily verified that $(H,C) \setminus \{a,b\}/\{c,d\}$ and $(H,C)/\{a,b\} \setminus \{c,d\}$ are related by a Lovász-flip (and possibly signature exchanges) and that any two signed graphs related by a single Lovász-flip can be obtained in that way. In particular, graphs related by Lovász-flips have the same set of even cycles.

3.3.3 A corresponding operation for even cuts

Let us now find a counterpart to the Lovász-flip operation for even cuts. Consider a graph G which consists of components $G[X_1]$ and $G[X_2]$, for some partition X_1, X_2 of $E(G)$. For $i = 1,2$, pick vertices s_i, t_i, u_i, v_i in $G[X_i]$. Denote by C the set of edges $\{a,b,c,d\}$, where $a = (s_1,s_2), b = (t_1,t_2), c = (u_1,u_2)$ and $d = (v_1,v_2)$. Let H be the graph obtained from G by adding the edges in C. Let $T := \{s_1,s_2,t_1,t_2,u_1,u_2,v_1,v_2\}$. Since C is an even cut of (H,T), it is a cycle of ecut(H,T). Moreover, C is a T-join of H. It follows from Remark 8 and Remark 16(2) that C is a cocycle of ecut(H,T). Lemma 18 implies that ecut$(H,T) \setminus \{a,b\}/\{c,d\} =$ ecut(H,T) $/\{a,b\} \setminus \{c,d\}$. It follows from (1) that ecut$\big((H,T) \setminus \{a,b\}/\{c,d\}\big) =$ ecut$\big((H,T)/\{a,b\} \setminus \{c,d\}\big)$. Hence, the two grafts $(H,T) \setminus \{a,b\}/\{c,d\}$ and $(H,T)/\{a,b\} \setminus \{c,d\}$ have the same even cuts. This defines a new operation that preserves even cuts. It turns out, however, that this operation is a special case of the Tilt operation introduced by Gerards [2] for even cycles.

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